Rational Homotopy Theory

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Location: SR 3

Feel free to join us!

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Homotopy theory is the study of topological spaces up to homotopy equivalence. The most important invariants in this subject are the homotopy groups $\pi_i(X)$. They are defined as the sets of homotopy classes of basepoint-preserving maps from the sphere S^i to the space X:

$$\pi_i(X, x_0) = [S^i, X]_*$$

For $i \geq 1$ they are indeed groups, for $i \geq 2$ even abelian groups, which carry a lot of information about the homotopy type of X. However, even for spaces which are easy to define (like spheres), they can be very hard to compute.
Even in low dimensions it is difficult to see a clear pattern among the homotopy groups of spheres; especially the torsion shows a seemingly wild behaviour. This suggests that in a first step it might be a good idea to ignore the torsion in the homotopy groups and to just consider the *rational homotopy groups* (the homotopy groups tensored with $\mathbb{Q}$).

Questions:

1. Given a space X, can we compute the rank of $\pi_i(X)$?

2. Can we classify topological spaces up to the torsion part of homotopy groups?

This is what rational homotopy theory is all about! In particular, it allows to give a wide understanding of rational homotopy groups of many spaces, as for example spheres.

Schedule

	Mo 1.4.	**Tue 2.4.**	**Wed 3.4.**
09:30 - 10:00	Introduction		
10:00 - 11:00	Homotopy I	Rat. Homotopy II	H-spaces
11:30 - 12:30	Homotopy II	Obstruction I	Loop Spaces
13:00 - 14:00	Lunch break	Lunch break	Lunch break
14:00 - 15:00	Simplicial Techniques	Obstruction II	Geodesics
15:30 - 16:30	Rat. Homotopy I	Formality of dga's	Kähler manifolds

List of talks

1. **Basics of Homotopy Theory I:** The purpose of the talk is to introduce/ recall some relevant notions from algebraic topology: the definition of homotopy groups, n-connectedness and the definition of CW-complexes. One could follow [**?**, §1, 2]. Define weak homotopy equivalences and state Whitehead's theorem in the homotopy ([**?**, 10.3, p.76]) and the homology version ([**?**, 4.2, p.367]) and recall Hurewicz theorem ([**?**, 4.2, p.366]). Make sure we understand the difference between
   - two spaces with the same homotopy groups
   - two weakly equivalent spaces (and the notion of weak homotopy type)
   - two homotopy equivalent spaces (and the notion of homotopy type).

2. **Basics of Homotopy Theory II:** Define Serre and Hurewicz fibrations. Explain the path space fibration [**?**, 2(b) Example 1], and fiber bundles [**?**, 2(d) Prop. 2.6] as examples. State/ Explain the long fiber sequence from [**?**, Thm. 8.6] and derive the long exact sequence of homotopy groups. This is the source for many of the long exact sequences in algebraic topology. Use it to express higher homotopy groups of $\mathbb{CP}^\infty$ (cf. [**?**, Example 4.50]).
   Define rational homotopy equivalences as *zigzag chains* (cf. [**?**, Prop. 9.8]), the notion of rational homotopy type and rationalizations of spaces. Show the explicit construction of the rationalization of the sphere [**?**, 1.1]. State Whitehead-Serre [**?**, Thm. 8.6]. If there is time left, one may explain how to rationalize CW-complexes.

3. **Simplicial Techniques:** Introduce simplicial objects (simplicial sets, simplicial graded algebras) following [**?**, 10(a)] and compare the two different points of view: as a functor and via explicit face and boundary morphisms. Define/ explain the adjunction between the singular set functor and geometric realization [**?**, I.2, Prop. 2.2]:
   $$Top \underset{|\cdot|}{\overset{Sing}{\rightleftarrows}} sSet$$
   and mention that it allows to phrase homotopy theory entirely in the language of simplicial sets. Given a simplicial set K, a simplicial cochain algebra B, introduce the "ordinary" cochain algebra $B(K)$ and its properties [**?**, 10(b)].
   Remind us how singular cohomology of a topological space X is defined in terms of the singular set $Sing(X)$ and the singular cochain algebra
   $$C^*_{sing}(X) := C^*(Sing(X))$$
   built out of $Sing(X)$. As an example consider the simplicial cochain algebra $B := C_{PL}$ as in [**?**, 10(d)] and mention that $C_{PL}(Sing(X)) \cong C^*_{sing}(X)$ are isomorphic as cochain algebras (cf. [**?**, Lemma 10.11]).
   Note that this construction extends to *any* simplicial set.

4. **Rational Homotopy Theory I:** We want to give an algebraic description of the rational homotopy category. This means we look for a rational homotopy invariant that is *sharp* in the sense that two spaces are rationally homotopy equivalent iff the invariants are the same.
   As another example of a cochain algebra B introduce the *commutative dg-algebra* A_{PL} and its elements, the *polynomial differential forms with coefficients in* $\mathbb{Q}$. Given a simplicial set K, state the existence of cochain *algebra* quasi-isomorphisms (as in [**?**, Cor. 10.10])
   $$C_{PL}(K) \longrightarrow (C_{PL} \otimes A_{PL})(K) \longleftarrow A_{PL}(K).$$
   Given X and Y simply connected topological spaces with same rational homotopy type (i.e. there exists a *zigzag chain* of rational homotopy equivalences), explain that $A_{PL}(X)$ and $A_{PL}(Y)$ are weakly equivalent. So A_{PL} is an invariant of rational homotopy type.
   Mention that integration gives a map $C_{PL} \longrightarrow A_{PL}$ which induces quasi-isomorphisms of *chain complexes* $C_{PL}(K) \longrightarrow A_{PL}(K)$ for all simplicial sets K.
   Explain the comparison theorem for a smooth manifold M relating the de Rham complex $\Omega^*_{dR}(M)$ to the complex $A_{PL}(M;\mathbb{R})$ [**?**, Thm. 11.4].

5. **Rational Homotopy Theory II:** In order to use the equivalence (defined in the previous talk) for concrete computation, introduce (minimal) Sullivan models [**?**, 12(a) below Example 5]. If time permits, indicate the construction of Sullivan models [**?**, Prop. 12.1]. One should present some simple computations as e.g. the rational homotopy groups of S^n, $\mathbb{CP}^n$ and $\mathbb{CP}^\infty$. (Remind us of $\mathbb{Z}/n\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Q} = 0$.) Explain how the rational homotopy groups can be read off from the minimal model [**?**, 13 (c) and 15.11].
   The crucial property of minimal Sullivan algebras is that every quasi-isomorphism between them is an isomorphism [**?**, 12.10]. Conclude that there is a bijection between rational homotopy types and minimal Sullivan algebras [**?**, 12].

6. **Intermezzo on obstruction theory:** Define the obstruction cochain and explain its basic properties [**?**, 6.2]. Elucidate the significance of the obstruction class to inductively construct homotopies between maps resp. sections of bundles [**?**, p.57-58]. (Another treatment can be found e.g. in [**?**, Chapter 7].) Mention that the obstruction class is well-defined and natural [**?**, p.58-59].
   Show how these concepts work in examples, e.g. how the Euler class is the first obstruction to construct a certain section [**?**, p.58-59] and classify the homotopy classes of maps into an Eilenberg McLane space $K(\pi, n)$ [**?**, p.63-64].
   If time permits, show how this classification allows us to identify the obstructions relevant to extending a section of a $K(\pi, n)$-fibration [**?**, p.64-66].

7. **Simply connected rational homotopy types and minimal dga's over $\mathbb{Q}$:** Explain [**?**, Chapter 12] how the inductive construction of *dg*-algebras via iterated Hirsch extensions [**?**, Chapter 10] corresponds to the construction of a space

through its Postnikov tower [?, Chapter 8] via iterated $K(\pi, n)$-fibrations.
Indicate how obstruction theory for dga's [?, Chapter 11] and spaces yields (under some finiteness assumptions) a bijection between simply connected rational homotopy types and minimal *dg*-algebras over $\mathbb{Q}$. Show how this bijection can be upgraded to an equivalence between the rational homotopy category of spaces and the homotopy category of *dg*-algebras [?, Chapter 15]. (Compare also [?, Chapter 17]).

8. **Formality of a dg-algebra:** Introduce the notion of formality (and non-formality) of a *dg*-algebra and state that it may be checked over any field extension [?, 12 (c), last theorem].
The idea of the talk should consist of examples to the notion of formality following [?, 12 (e) Example 1-3]:

    - Discuss the relation between Lie-algebra cohomology and the minimal model of a Lie group.
    - Introduce nilmanifolds and show they are almost never formal.
    - Introduce symmetric spaces and show that they are formal.

9. **Example on H-spaces:** This talk should determine the rational homotopy type of H-spaces. Introduce H-spaces. H-spaces are spaces with a product-up-to-homotopy structure, which includes topological groups and loop spaces. The existence of an H-space is a strong condition on the homotopy type.
Prove that the minimal model of an H-space is an exterior algebra with zero differential, following [?, 12(a)], or explain that the cohomology of an H-space is a Hopf algebra and state the classification of finite dimensional Hopf algebras over the rational numbers (cf. [?, Appendix]), which gives the same result.
Conclude that H-spaces are always formal and that Lie groups have the rational homotopy type of a wedge of odd spheres.

10. **Loop spaces:** This talk focuses on loop spaces and their models in rational homotopy theory. Explain that the minimal model of a loop space is an exterior algebra with vanishing differential. We would like to know the number and degree of its generators.

    - Explain how models behave under fibrations [?, Thm 15.3] and [?, Thm 2.2].
    - Apply this to the path space fibration to compute the minimal model of loop spaces [?, Ex 2.3].

    Explain how models behave under pullback [?, 15(c)] and apply to compute the Sullivan model of a free loop space [?, 15(c), Ex. 1].

11. **Geodesics:** In this talk, we want to study Morse theory on the free loop space over a (simply connected, compact) smooth manifold and derive insights about

geodesics on the manifold from the rational homotopy theory.
In particular we want to understand geodesics as critical points of the geodesic action functional on the free loop space, how the number of critical points is related to Betti numbers and how one can use this to show that there are infinitely many geometrically distinct closed geodesics on simply connected, compact Riemannian manifolds which need at least two generators for their rational cohomology algebra. The geometric part of this theory was done by Gromoll and Meyer in [**?**] and the rational homotopy part by Vigué-Poirrier and Sullivan in [**?**].

12. **Formality of Kähler manifolds:** We conclude the seminar with the proof of [**?**] that every Kähler manifold is formal and deduce as a corollary that there exist symplectic manifolds which are not Kähler. We want to understand the dd^c-lemma on Kähler manifolds and how it implies formality.
    There is also the interesting fact that this proof fails for generalized complex manifolds, even if they admit a dd^c-lemma [**?**].