RTG Seminar - Topological K-Theory

October 19–22 2020

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Outline

Topological K-Theory is similar to a cohomological theory in the construction. It associates to every pair of compact topological spaces (X, Y) a graded family of abelian rings $K^n(X, Y)$. It can be viewed as a way to classify vector bundles on compact topological spaces (up to an equivalence relation coarser than isomorphism). K-groups are associated to topological spaces using the so-called Bott periodicity, making the whole theory computable. Applications of topological K-theory include the classification of real division algebras, and the solution of the question of how many linearly independent vector fields exist on a sphere.
Our seminar follows the first two chapters of the classical book *K-Theory* by Atiyah. First, we recall the notion of vector bundles over compact spaces and all the associated operations. This will lead us to the definition of the K-group $K(X)$ associated to a compact topological space X and then that of a pair (X, Y). We will then cover Bott periodicity, referred by Atiyah as the fundamental theorem of K-theory, which allows us to extend the functor K to a graded version K^n for every integer n. We then deal with the cohomological properties of the resulting K-theory and compute it for some basic spaces. At the end of the seminar we cover various applications.

References

The previous plan is strongly inspired from a seminar of T. Vogel (LMU) [1]. The main reference is the classical book *K-Theory* by Atiyah (Advanced Book Classics, 1967).

Program

Talks are one hour. Half an hour break is programmed after each talk to allow for questions, further discussions and coffee.

	Oct 19	Oct 20	Oct 21	Oct 22
09:30-10:30		Vect. Bundles III	Bott period. I	Multiplication
11:00-12:00	Vect. Bundles I	K-Theory I	Bott period. II	Thom-Künneth
14:00-15:00	Vect. Bundles II	K-Theory II	Examples	Application I

[1] `http://www.math.lmu.de/~tvogel/Seminare/Ktheoryws18-19/KtheoryProg.pdf`

List of talks

1. **Vector bundles I - Introduction** (§1.1-1.3) Introduce vector bundles on topological spaces and the associated notions (sections, morphism of bundles,...). List and explain the main operations on vector bundles (eg. direct sum, tensor product). Finally explain the notions of sub-bundle and quotient bundle. You may consider using other sources for this talk.

2. **Vector bundles II - Compact bases** (§1.4 until Lemma 1.4.6 (which is after Lemma 1.4.8) incl.) What is special about vector bundles over a compact base ? Present the results about homotopy invariance for (para)compact base spaces. Introduce the abelian semi-groups $\mathrm{Vect}(X)$ and $\mathrm{Vect}_n(X)$. Discuss the "collapsing" and "clutching" operations for vector bundles (see Remark after Lemma 1.4.6).

3. **Vector bundles III - Homotopy theoretic approach** (§1.4 from Lemma 1.4.9) Start with the homotopy theoretic interpretation of $\mathrm{Vect}_n(S(X))$. Proceed with the homotopy theoretic interpretation of $\mathrm{Vect}(X)$ via Grassmannians. If time permits, you can present selected topics from §1.5.

4. **K-Theory I - Definition of $K^{-n}(X)$ and $K^{-n}(X, Y)$** (§2.1 and §2.2 until Prop. 2.2.5 excl.) Define the functor K of spaces. Give its homotopy theoretic interpretation in the compact case. Define the functors K^{-n} for compact spaces and pairs of spaces using the cone construction. Prove the exactness of the sequence of Lemma 2.2.3.

5. **K-Theory II - Long exact sequence** (Prop 2.2.5 and the end of §2.2) State and prove the long exact sequence that relates the functors K^{-n}.

6.-7. **Bott periodicity**(§2.3 until p. 74) These two talks should cover the Bott periodicity Theorem and its connection to K-Theory. Namely, prove that $K(X) \cong K^{-2}(X)$. The two speakers are expected to organise themselves and decide how to split the work. One can either follow Atiyah's approach or Bott's original treatment using Morse theory. A good reference for the latter is the eponymous set of notes by Milnor.

8. **Examples and cohomological interpretation** (§2.3 after p.75 and §2.5) Compute $K(\text{point})$ and $K(\mathbb{P}(L \oplus 1))$ for a line-bundle L. Explain how to use the previous computations (and hence Bott Periodicity) to make K-theory into an (almost) cohomology theory. Compute $K(\mathbb{CP}^n)$ and the other examples of §2.5.

9. **Multiplication** (§2.6) Discuss the multiplication operation in $K^*(X, Y)$. Then proceed with the discussion of the Euler characteristic in this context. You may chose to present only selected topics about the Euler characteristic.

10. **Thom Isomorphism and Künneth Theorem** (§2.7 without the digression from Cor 2.7.3 to Prop 2.7.7 incl.) Recall the notion of Thom space associated to

a topological space. Explain how $\widetilde{K}^*(X^E)$, where X^E is the Thom space of X, enjoys the structure of a $K(X)$-module. State and discuss the Splitting Principle. If time permits, discuss the Künneth Theorem.

11. **Application I - Stable homotopy groups of spheres** Introduce the notions of Chern character and J-homomorphism. Explain how K-Theory can be applied to the study of stable homotopy groups of spheres. This is a problem we mentioned in our Rational Homotopy Seminar last year. A reference could be *Vector bundles and K-Theory* by Hatcher.