RTG Seminar - Holomorphic dynamics à la Milnor

March 8-9 & 11-12 2021

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Outline

A *dynamical system*, in the most abstract sense, is a continuous map $f\colon X \to X$ from a metric space X to itself. The space X is partitioned into *orbits* where two points of X are in the same orbit if one can be mapped to the other by an iteration of f. Studying a dynamical system essentially means understanding its **orbit structure**. In this seminar we are interested in the class of dynamical systems that consist of holomorphic functions $f\colon S \to S$ from a Riemann surface S to itself.
Any holomorphic map $f\colon S \to S$ induces a partition of the Riemann surface S into its so-called *Julia set* and its complement, the *Fatou set*. The goal of the seminar is to understand the structure of these two remarkable sets. The Julia set $J(f)$ of $f\colon S \to S$ is, broadly speaking,

$$J(f) := \{z \in S : \text{the dynamics of } f \text{ near } z \text{ is } \textit{wild}\}.$$

The Fatou set, defined as the complement of the Julia set, is the collection of points where the dynamics is *tame*. The Julia set is closed and invariant under f. It is in general singular and tends to behave like a **fractal**.
We will essentially consider the case where $S = \widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ is the Riemann sphere. This allows for illustrations of the Julia set, such as on the picture below.

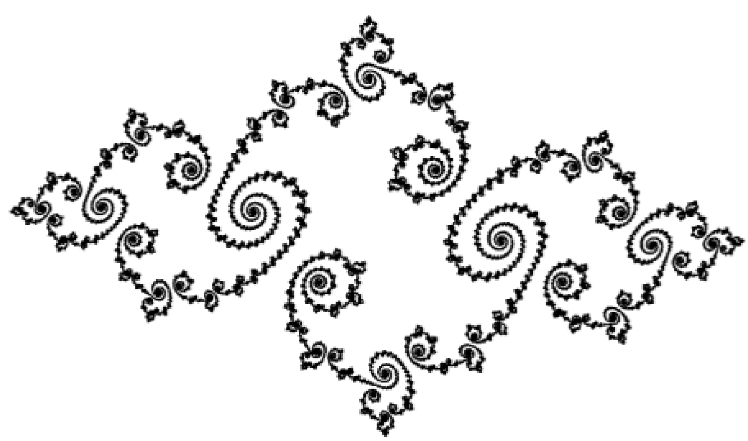

Figure 1b. A totally disconnected Julia set,
$z \mapsto z^2 + (-.765 + .12i)$.

We will study the different kind of fixed points of a holomorphic map $f\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$. These include *attracting and repelling fixed points*, as well as *parabolic fixed points*. Among different results, we will prove that almost every orbit of a rational map is *repelling* and that the Julia set is equal to the closure of the set of repelling periodic points. The seminar will end with a discussion of the classification by Sullivan of the components of the Fatou set.

This seminar is at the boarder between the theories of **dynamical systems** and **Riemann surfaces**. It requires no particular familiarity with either of the two topics. We will start by recalling some fundamental results on Riemann surfaces. We will introduce the relevant notions of dynamical systems along the way (e.g. periodic and fixed points, invariant sets, attracting and repelling points).

The main reference for the seminar are the lectures notes [Mil06] by Milnor. The numbering below always refers to the arXiv version of the notes. For any further reference on the topic of dynamical systems we will use [KaHa95]. The lecture notes [Rug18] by Ruggerio will supplement the main reference.

References

[Mil06] Milnor, John. Dynamics in one complex variable. Third edition. Annals of Mathematics Studies, 160. Princeton University Press, Princeton, NJ, 2006. viii+304 pp. ISBN: 978-0-691-12488-9; 0-691-12488-4. ArXiv version: `https://arxiv.org/pdf/math/9201272.pdf`

[KaHa95] Katok, Anatole; Hasselblatt, Boris. Introduction to the modern theory of dynamical systems. Encyclopedia of Mathematics and its Applications, 54. Cambridge University Press, Cambridge, 1995. xviii+802 pp. ISBN: 0-521-34187-6.

[Rug18] Ruggiero, Matteow. Holomorphic dynamical systems. Lecture notes, 2018. `https://webusers.imj-prg.fr/~matteo.ruggiero/teach/2018-19%201A/2018-19%201A-en.html`

Program

Each talk should last about one hour. Half an hour break is programmed after each talk to allow for questions, further discussions and coffee. The seminar happens online, via Zoom.

	Mon, Mar 8	Tue, Mar 9	Thu, Mar 11	Fri, Mar 12
09:00-09:30	Introduction			
09:30-10:30	Riem. surf. I	The sphere	Parabolic	Repelling orbits
11:00-12:00	Riem. surf. II	The others	Cremer & Siegel	Herman rings
13:30-14:30	Julia & Fatou			
14:00-15:00		Attrac. & Rep.	Index formula	Classification

List of talks

1. **Riemann surfaces I** (§1)

   The goal of this first talk is to recall some central notions about Riemann surfaces. After recalling their definition, present the statement of the Uniformization Theorem. Recall us of the Schwarz Lemma and the Theorem of Weierstrass. These will be useful tools later. The second part of the talk is dedicated to the properties of the upper-half plane and the Riemann sphere (group of automorphisms, natural metric) as two of the three models for simply connected Riemann surface.

   Complex geometry - 3/3 *Dynamical systems - 0/3*

2. **Riemann surfaces II** (§2)

   Explain how Riemann surfaces can always be written as the quotient of one of the three simply connected models by a discrete group. Discuss briefly each of three resulting families, mentioning the interesting examples, such as the 3-punctured sphere (and Picard Theorem). The second part of the talk introduces the central notion (for this seminar) of *normal* family of holomorphic maps. Make sure we understand this notion. State and prove Montel's Theorem.

   Complex geometry - 3/3 *Dynamical systems - 0/3*

3. **Definition of Julia and Fatou sets** (beginning of §3 & §5)

   This talks sets the foundations of the seminar. Define the notions of Julia and Fatou sets for a holomorphic map of a Riemann surface. Explain their basic properties (Lemmata 3.1 & 3.2) while introducing the relevant notions from dynamical systems theory (e.g. orbit, grand orbit, invariance). Discuss the smooth examples of §5 and show some illustrations of your favourite examples.

   Complex geometry - 0/3 *Dynamical systems - 3/3*

4. **The sphere** (end of §3)

   Continue with the properties of Julia sets in §3 (Theorem 3.3 and onwards). On the way, introduce the relevant notions from dynamical systems theory (e.g. periodic orbits, attracting and repelling sets, basin of attraction). Proceed with the case of the Riemann sphere. Recall why holomorphic maps of the Riemann sphere are always quotients of polynomials (you might need to find a source for this). Discuss the properties of the associated Julia sets (Lemma 3.5 and onwards).

   Complex geometry - 1/3 *Dynamical systems - 2/3*

5. **The others** (§4)

   Start by explaining why studying Julia sets is dull in the case of the torus. Proceed with the case of hyperbolic Riemann surfaces. Prove the Denjoy-Wolff Theorem (Theorem 4.2) and discuss the statement of Theorem 4.3. The latter will be useful for the last talk. You can mention briefly the cases of the complex plane and the punctured plane.

Complex geometry - 2/3 *Dynamical systems - 1/3*

6. **Attracting and repelling fixed points** (§6)

   State and prove the Koenigs Theorem. Explain what this theorem implies about the structure of Julia sets (Remark 6.2). Discuss the consequences for attracting and repelling fixed points. State and prove Theorem 6.6, originally due to Julia and Fatou. Conclude the talk by discussing the superattracting case. Don't forget to provide illustrations.

   Complex geometry - 0/3 *Dynamical systems - 3/3*

7. **Parabolic fixed points** (§7)

   This is a talk about petals and flowers – green-thumbed speaker only. Introduce the notion of petals and explain the statement of the Flower Theorem. State its various corollaries and prove the theorem in case $\lambda = 1$. Don't forget the pictures. Explain how to generalize the results for any root of unity (Lemma 7.6). You can ignore the rest of §7.

   Complex geometry - 0/3 *Dynamical systems - 3/3*

8. **Cremer points and Siegel disks** (§8)

   This chapter discusses the case when λ has modulus one but is not a root of unity. Explain the centre problem. Explain the results known about this problem (at least when the notes were written). Define the notions of Cremer/Siegel points and Cremer/Siegel disks. Prove Lemma 8.1. You can briefly give an overview of the rest of §8. The lecture notes [Rug18] may help.

   Complex geometry - 0/3 *Dynamical systems - 3/3*

9. **Index formula** (§9)

   Introduce the notion of holomorphic index of a fixed point. Prove the Holomorphic Fixed Point Formula and prove that the index is a local analytic invariant. Conclude that the Julia set of a rational map of degree at least two is nonempty. You might want to recall some results from complex analysis along the way.

   Complex geometry - 2/3 *Dynamical systems - 1/3*

10. **Repelling orbits** (§10 & §11)

    The first part of the talk concerns §10. The goal is to understand why almost every periodic orbits are repelling. The proof splits up into smaller lemmata. The second part of the talk is dedicated to the proof that the Julia set is actually the closure of the set of repelling periodic points. You can present the proof by Julia. It makes use of the notion of homoclinic orbits.

    Complex geometry - 0/3 *Dynamical systems - 3/3*

11. **Herman rings** (§12)

    Define the notion of Herman ring. We want to understand what conditions imply the existence of a Herman ring. It is worth introducing the concept of rotation numbers for circle hoemomorphisms in more details than provided in §12. One could have a look at [KaHa95] for that. Recal what Diophantine numbers are and state Herman-Yoccoz Theorem. Define Blaschke products and explain how there are related to rational maps. Discuss the existence of Herman rings for Blaschke products. Feel free to show some pictures.

    Complex geometry - 0/3   *Dynamical systems - 3/3*

12. **Sullivan classification** (§13)

    We end the seminar with the classification of Fatou components by Sullivan. Start by explaining the classification for invariant Fatou components. The proof is based on the so-called Snail Lemma. Explain how to extend the classification to components which cycle periodically. Conclude the classification by stating Sullivan Theorem which prohibits wandering Fatou components.

    Complex geometry - 0/3   *Dynamical systems - 3/3*