The two natures of components of character varieties

Arnaud Maret
Sorbonne Université

Topology Seminar
University of Iowa

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(1) Character varieties

oriented
(closed)
surface

Lie group
(e.g. $PSL_2\mathbb{R}$)

$$\mathrm{Hom}(\pi_1(\Sigma_g), G)$$

$$\mathrm{Hom}(\pi_1(\Sigma_g), G) \curvearrowleft \mathrm{Inn}(G)$$

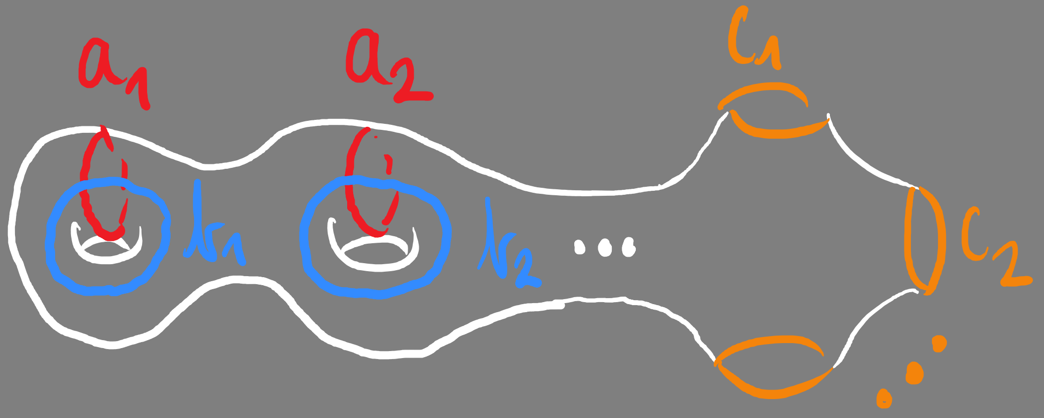
$$\begin{aligned} G &\longrightarrow G \\ h &\longmapsto ghg^{-1} \end{aligned}$$

$$\mathrm{Rep}(\Sigma_g, G) := \mathrm{Hom}(\pi_1(\Sigma_g), G) \,/\, \mathrm{Inn}(G)$$

character variety of (Σ_g, G)

oriented punctured surface

$$\mathrm{Hom}(\pi_1(\Sigma_{g,n}), G)$$

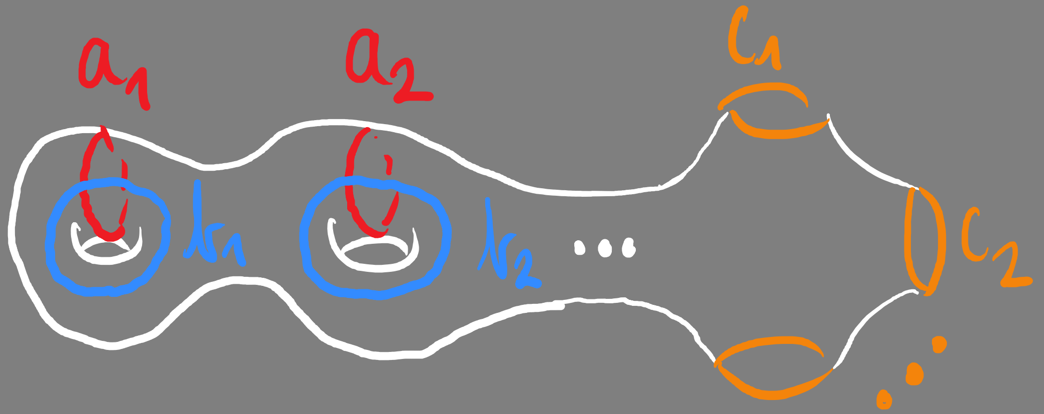
$\pi_1\Big($ 

$\Big)$

oriented punctured surface

$$\mathrm{Hom}\big(\pi_1(\Sigma_{g,n}), G\big)$$

$\pi_1\Big($

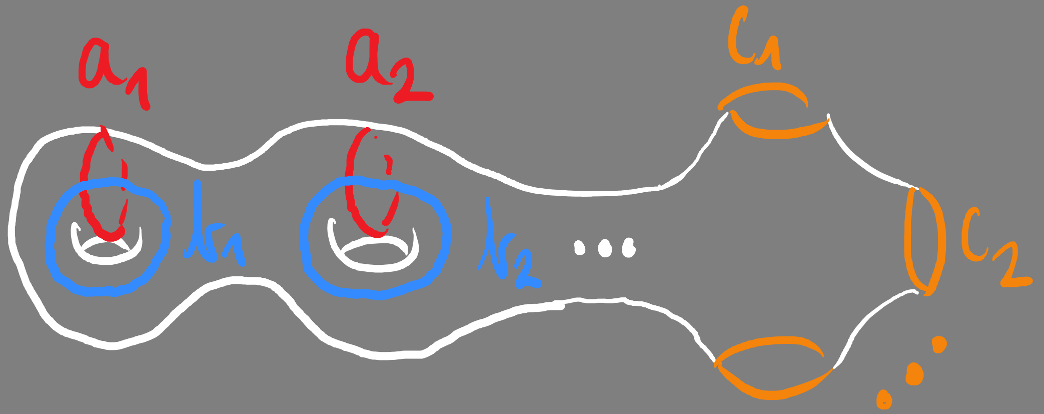


$\Big) \cong \left\langle a_i, b_i, c_i : \prod_{i=1}^{g} [a_i, b_i] = \prod_{i=1}^{n} c_i \right\rangle$

oriented punctured surface

$$\mathrm{Hom}(\pi_1(\Sigma_{g,n}), G)$$

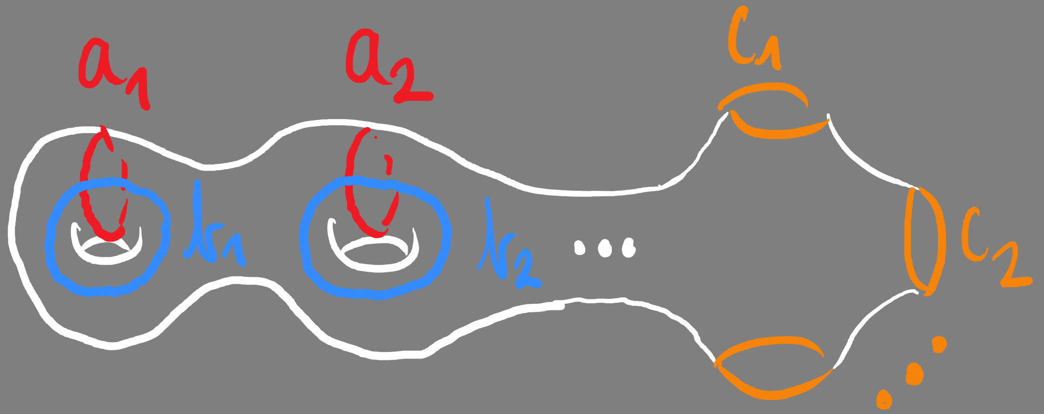
$\pi_1\Big($



$\Big) \cong F_{2g+n-1}$

oriented
punctured
surface

$$\mathrm{Hom}(\pi_1(\Sigma_{g,n}), G) \cong G^{2g+n-1}$$

$$\pi_1\Big($$  $$\Big) \cong F_{2g+n-1}$$

$$\mathrm{Hom}_{\mathcal{C}}(\pi_1(\Sigma_{g,n}), G) \leftarrow \phi(c_i) \in C_i$$

collection of n conjugacy classes

$$C_1, \dots, C_n \in G/\mathrm{conj}.$$

$$\mathrm{Rep}_{\mathcal{C}}(\pi_1(\Sigma_{g,n}), G) := \mathrm{Hom}_{\mathcal{C}}(\pi_1(\Sigma_{g,n}), G) \big/ \mathrm{Inn}(G)$$

relative character variety of $(\Sigma_{g,n}, G)$

Take-away #1

$\mathrm{Rep}_{\mathcal{C}}(\pi_1(\Sigma_{g,n}), G)$ is naturally symplectic

Take-away #1

$\exists$ closed non-degenerate 2-form

$\rightsquigarrow$ measure size of 2-dimensional objects

$\mathrm{Rep}_{\mathcal{C}}(\pi_1(\Sigma_{g,n}), G)$ is naturally symplectic

Take-away # 1

$\mathrm{Rep}_{\mathcal{C}}(\pi_1(\Sigma_{g,n}), G)$ is naturally symplectic

(Goldman, Atiyah-Bott, Karshon, Guruprasad-Huebschmann-Jeffrey-Weinstein, Lawton, ...)

Take-away #2

$$\mathrm{Aut}^{*}(\pi_1(\Sigma_{g,n})) \Big/ \mathrm{Inn}(\pi_1(\Sigma_{g,n})) \circlearrowleft \mathrm{Rep}_{\mathcal{C}}(\pi_1(\Sigma_{g,n}), G)$$

Take-away #2

$$\mathrm{Out}^*(\pi_1(\Sigma_{g,n})) \circlearrowright \mathrm{Rep}_{\mathcal{C}}(\pi_1(\Sigma_{g,n}), G)$$

Take-away #2

$$\begin{array}{ccc} & \mathrm{Out}^*(\pi_1(\Sigma_{g,n})) \hookrightarrow \mathrm{Rep}_{\mathcal{C}}(\pi_1(\Sigma_{g,n}), G) \\ (\text{index } 2) \longrightarrow & \vee & \\ & \mathrm{PMod}(\Sigma_{g,n}) & \end{array}$$

$\nwarrow$ pure mapping class group of $\Sigma_{g,n}$

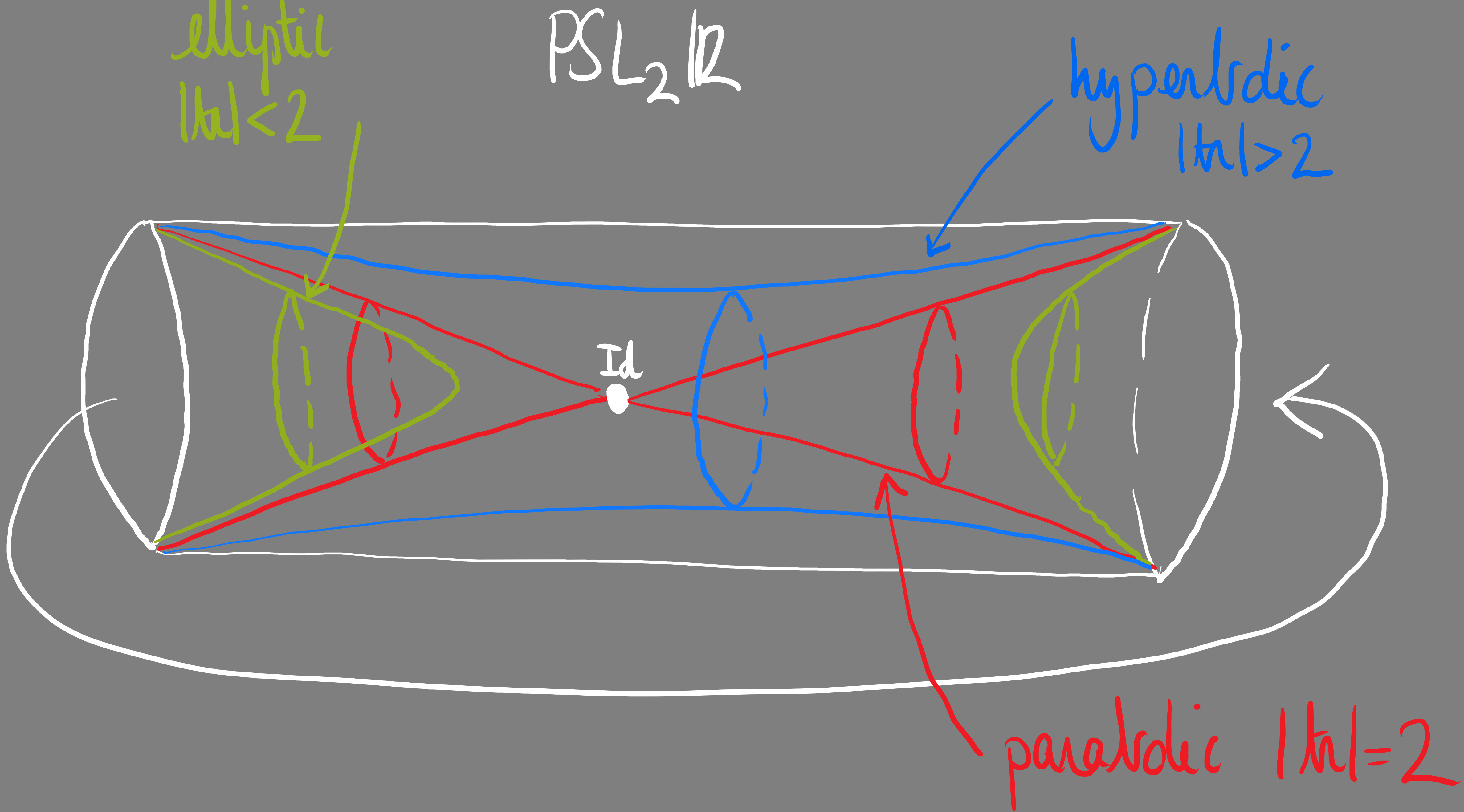
Take-away #2

$$\mathrm{PMod}(\Sigma_{g,n}) \curvearrowright \mathrm{Rep}_{\mathcal{C}}(\pi_1(\Sigma_{g,n}), G)$$

and the action is symplectic

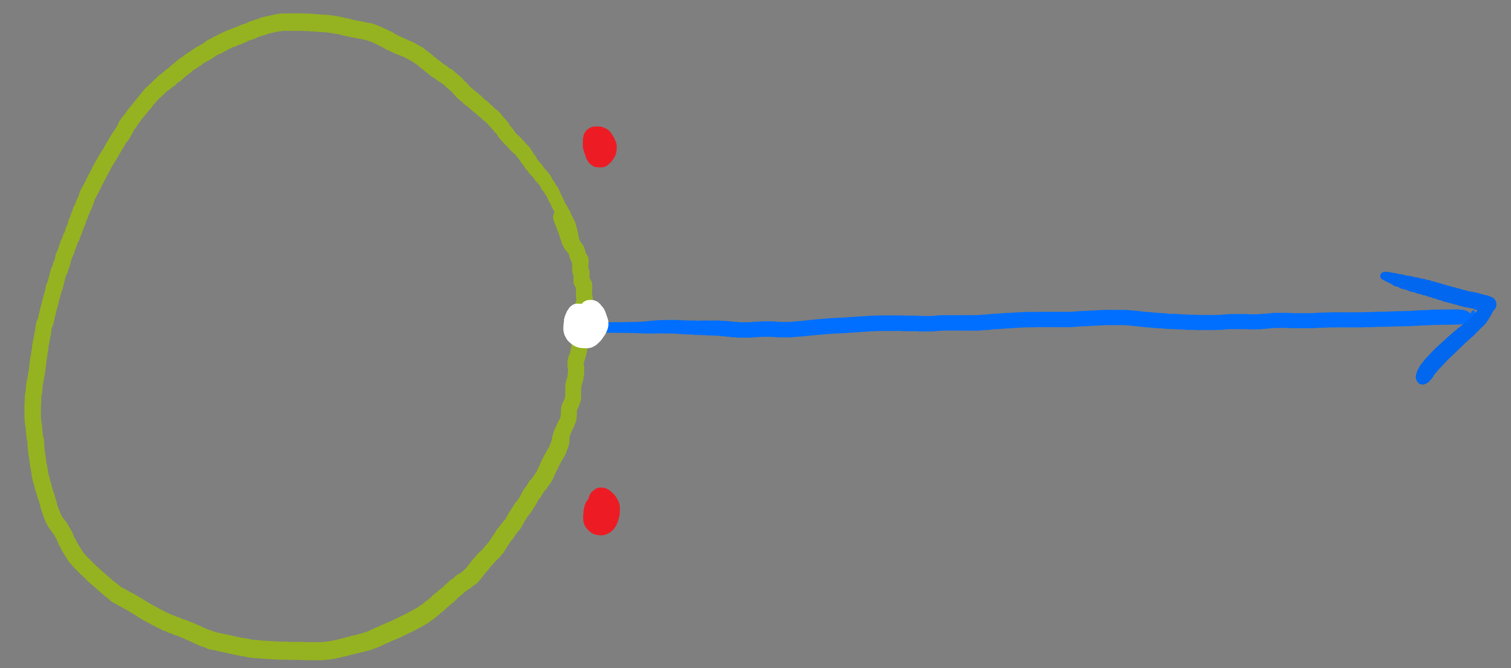
(2) Bowditch-Goldman program

$$\begin{tikzcd}[column sep=small]
 & PSL_2\mathbb{R} \arrow[dl] \arrow[dr] & & \{\det = 1\} \arrow[dl] \\
\mathrm{Isom}^+(\mathbb{H}^2) & & SL_2\mathbb{R}/\pm I &
\end{tikzcd}$$



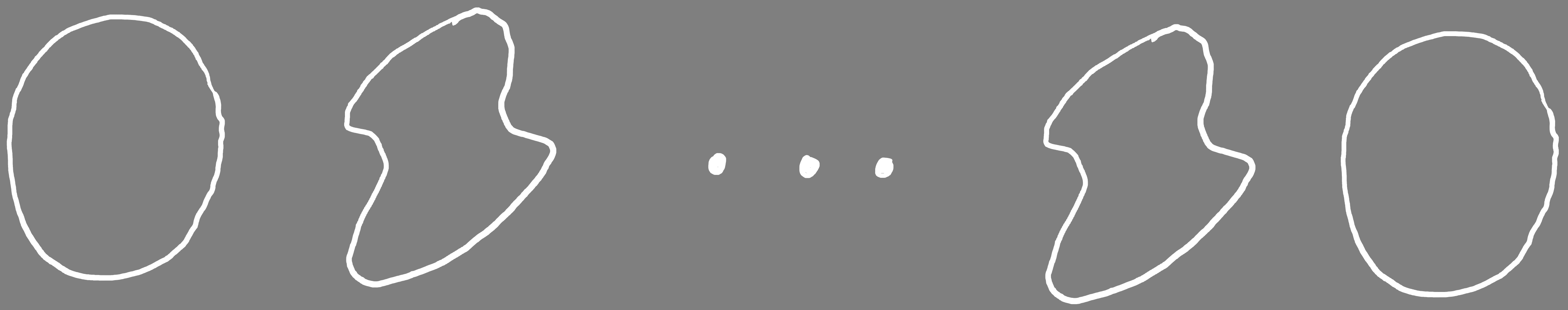
PSL2 R
elliptic |tr|<2
hyperbolic |tr|>2
Id
parabolic |tr|=2

$PSL_2\mathbb{R}$



$$\mathrm{Rep}(\Sigma_g, PSL_2\mathbb{R})$$

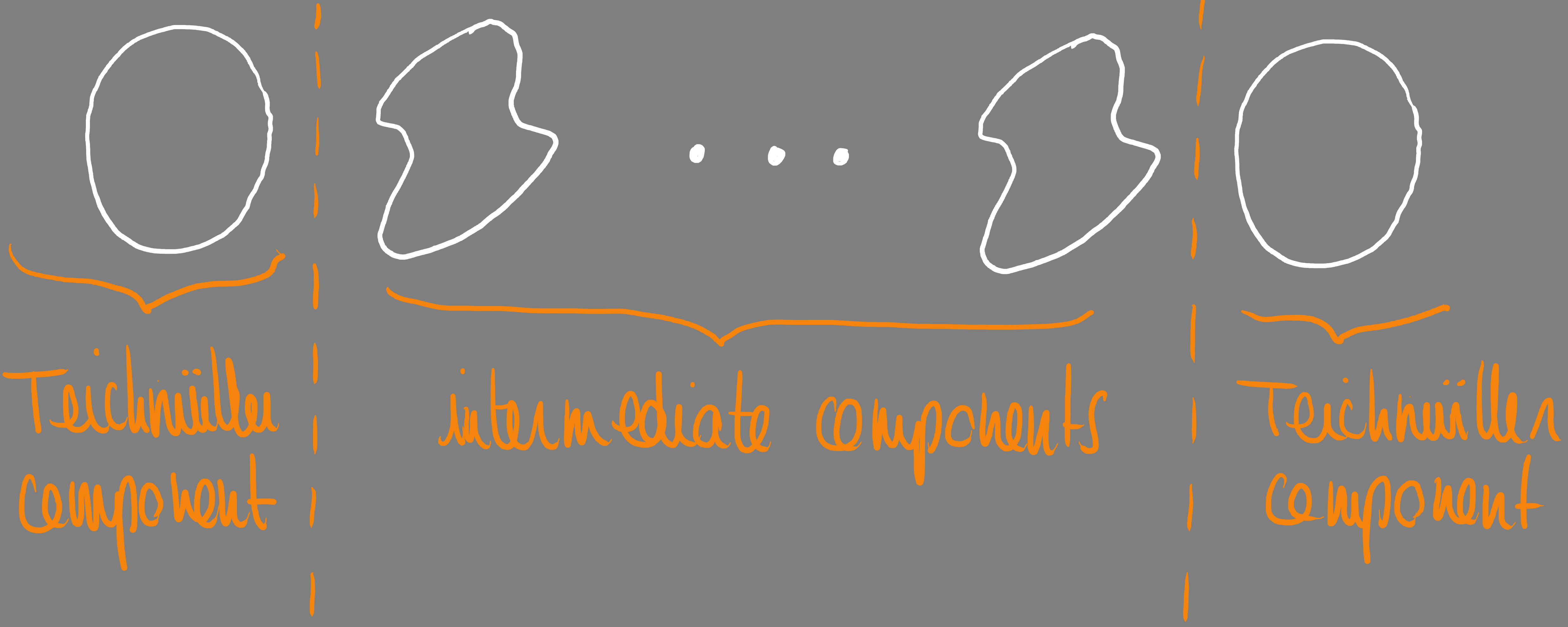
$$\parallel$$



$-2g+2$ $\quad$ $-2g+3$ $\quad$ $\cdots$ $\quad$ $2g-1$ $\quad$ $2g-2$

(Goldman 84')

$$\mathrm{Rep}(\pi_1(\Sigma_g), PSL_2\mathbb{R})$$



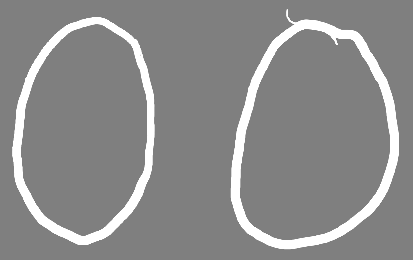
Type #1 : Teichmüller components

Mod-action

action is proper

$\phi : \pi_1(\Sigma_g) \to PSL_2\mathbb{R}$

* $\phi(\gamma)$ is hyperbolic $\forall \gamma \in \pi_1(\Sigma_g)$
* ϕ is discrete and faithful

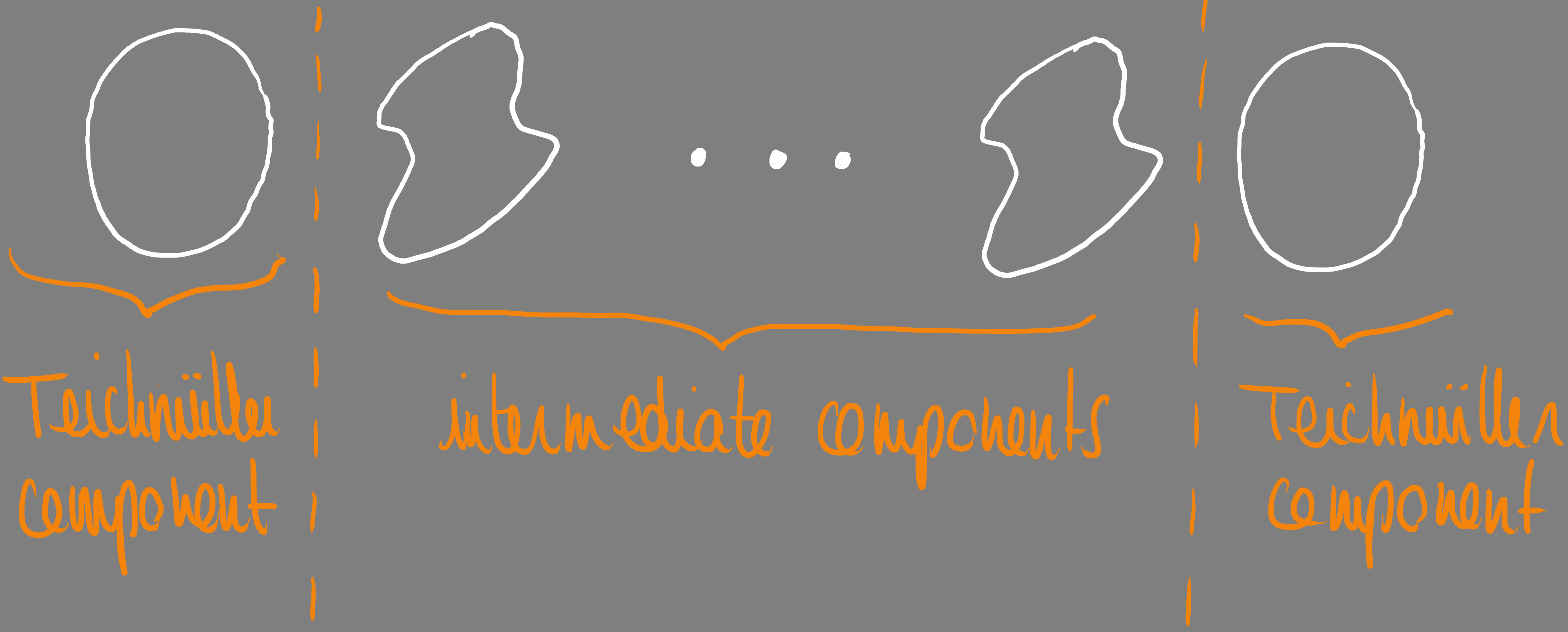


geometrization

$\bigcirc \cong \mathrm{Teich}(\Sigma_g)$

moduli space of marked hyperbolic structures on Σ_g

$\mathrm{Rep}(\pi_1(\Sigma_g), PSL_2\mathbb{R})$



Type #2 : Intermediate components

Bowditch's question (94')

$\exists$ simple closed curve γ

$\phi(\gamma)$ is non-hyperbolic

Goldman's conjecture (06')

$\mathrm{Mod}(\Sigma_g) \curvearrowright$ 

is ergodic

Geometrization conjecture

monodromies of branched hyperbolic structures

Bowditch-Goldman program

Bowditch's question

$\exists$ simple closed curve γ

$\phi(\gamma)$ is non-hyperbolic

Goldman's conjecture

$\mathrm{Mod}(\Sigma_g) \curvearrowright$

is ergodic

Geometrization conjecture

monodromies of branched hyperbolic structures

Some progress...

Bowditch's question

$\exists$ simple closed curve γ

$\phi(\gamma)$ is non-hyperbolic

Geometrization conjecture

monodromies of branched hyperbolic structures

Goldman's conjecture

$\mathrm{Mod}(\Sigma_g) \curvearrowright$ ☇ is ergodic

(Marché-Wolff 18')

Some progress...

Bowditch's question

$\exists$ simple closed curve γ

$\phi(\gamma)$ is non-hyperbolic

$g=2$

(Marché-Wolff 18')

Goldman's conjecture

$\mathrm{Mod}(\Sigma_g) \curvearrowright \mathcal{E}$

is ergodic

co-Euler

class ± 1

(Deroin 23')

Geometrization conjecture

monodromies

of branched

hyperbolic structures

obstacles ?

Bowditch's question

$\exists$ simple closed curve γ

$\phi(\gamma)$ is non-hyperbolic

Poincaré conjecture \ Perelman Theorem

$$\pi_1(\Sigma_g) \xrightarrow{\varphi} \!\!\!\to F_g \times F_g$$

$\Rightarrow$ $\exists$ simple closed curve in $\mathrm{Ker}(\varphi)$

(Stallings 65')

(3) Special case in genus zero

$$\mathrm{Rep}_{\alpha}(\Sigma_n, PSL_2\mathbb{R})$$

$\alpha = (\alpha_1, \ldots, \alpha_n)$

$\alpha_i \in (0, 2\pi)$

n elliptic classes

sphere with

$n \geq 3$ punctures

$$\mathrm{Rep}_\alpha(\Sigma_n, \mathrm{PSL}_2\mathbb{R})$$

$\alpha = (\alpha_1, \dots, \alpha_n)$

$\alpha_i \in (0, 2\pi)$

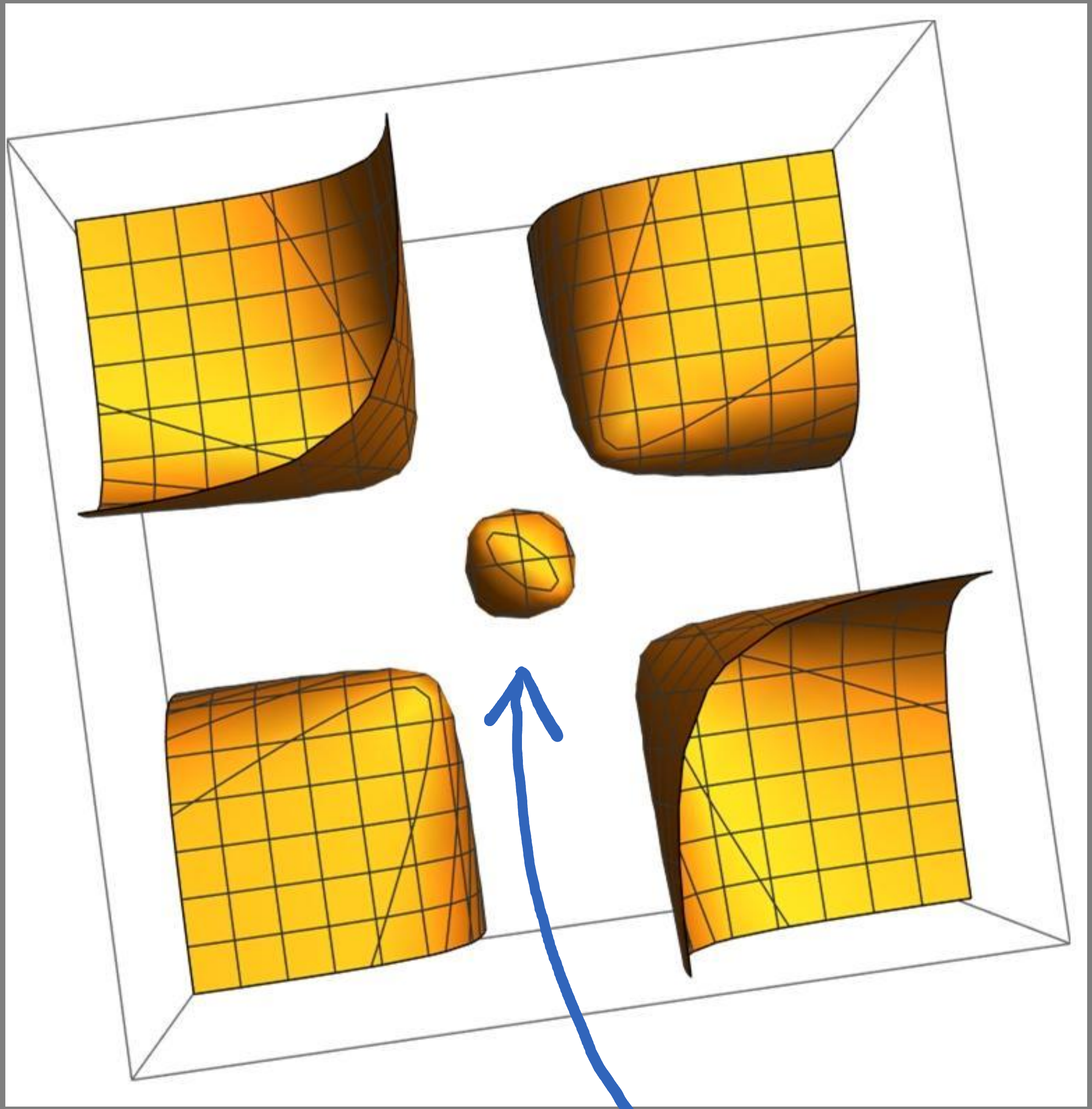
n elliptic classes

sphere with $n \geq 3$ punctures

<u>THM</u> (Deroin–Tholozan 20')

$$\alpha_1 + \dots + \alpha_n > 2\pi(n-1)$$

$\implies \exists$ compact component $\mathrm{Rep}^{DT}_\alpha \subseteq \mathrm{Rep}_\alpha(\Sigma_n, \mathrm{PSL}_2\mathbb{R})$



$n=4$

(Benedetto - Goldman 91')

$$\mathrm{Rep}^{DT}_{\alpha} \subseteq \mathrm{Rep}_{\alpha}(\Sigma_4, PSL_2\mathbb{R})$$

(Deroin–Tholozan 22')

$\phi \in \mathrm{Rep}^{DT}_{\alpha}$

$\Rightarrow \phi(\gamma)$ elliptic $\forall \gamma$

simple closed curve

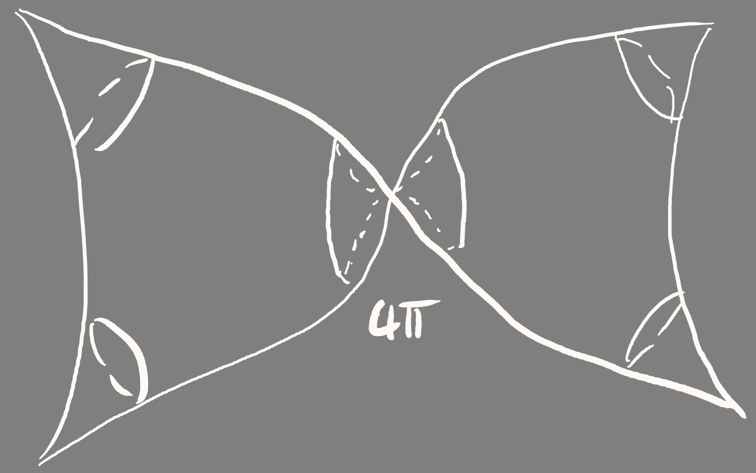
" totally elliptic "

(Deroin–Tholozan 22', M.–Fenyes)

monodromies of

branched conical

hyperbolic metric on Σ_n

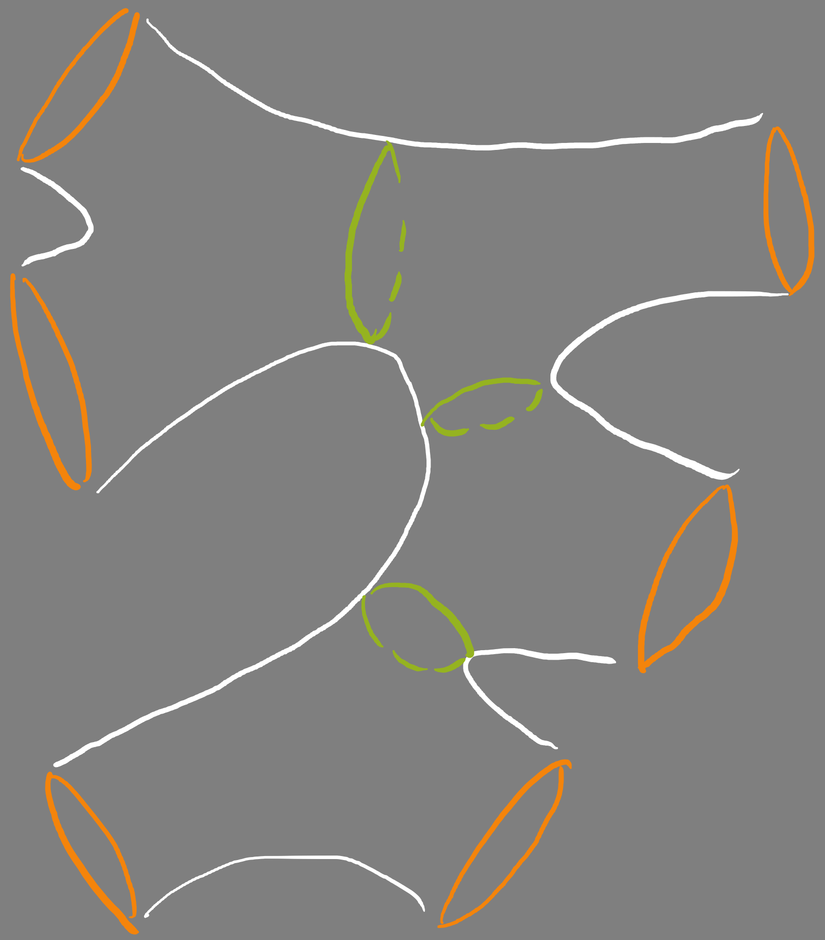


(M. 22')

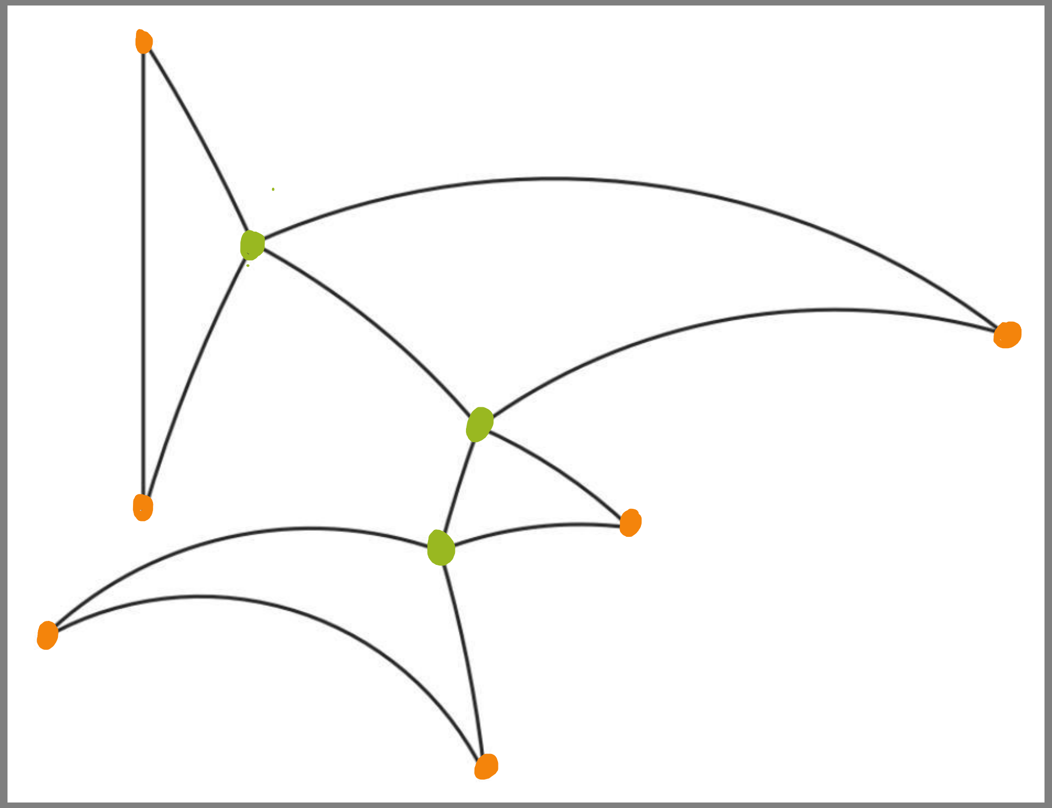
$\mathrm{PMod}(\Sigma_n) \curvearrowright \mathrm{Rep}^{DT}_{\alpha}$

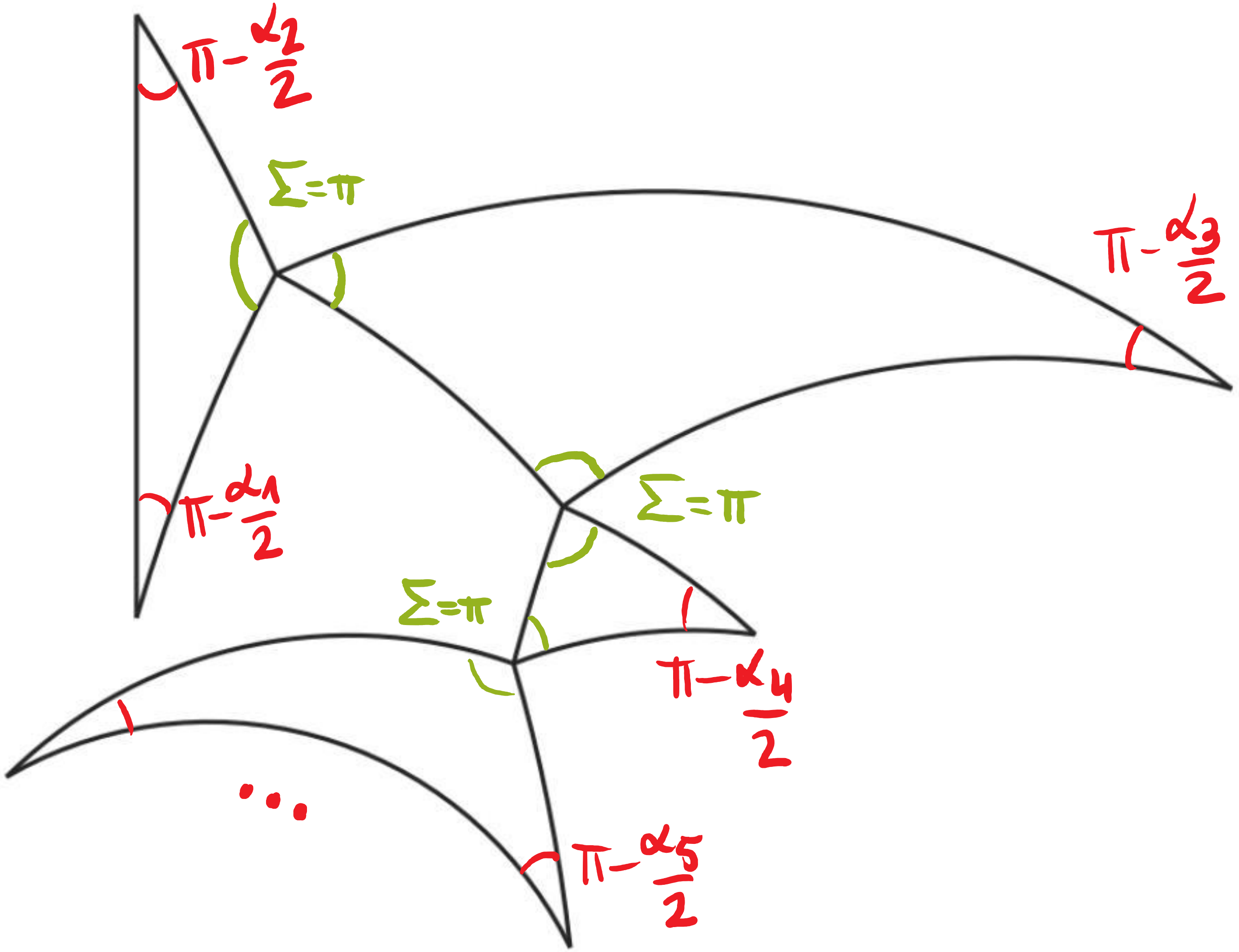
is ergodic

Σ_n

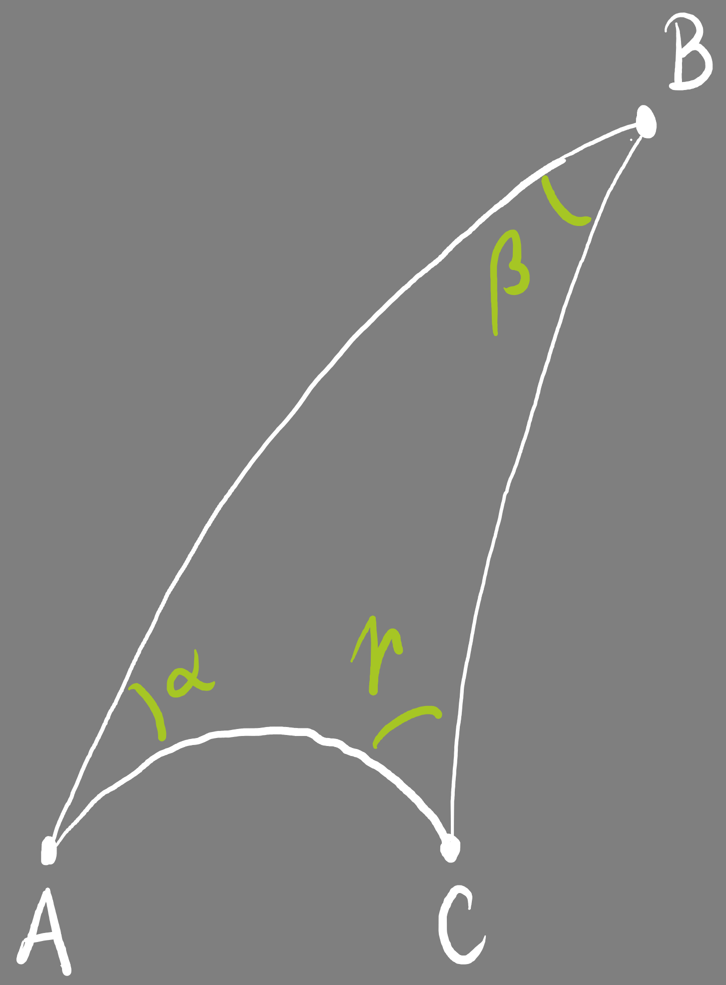


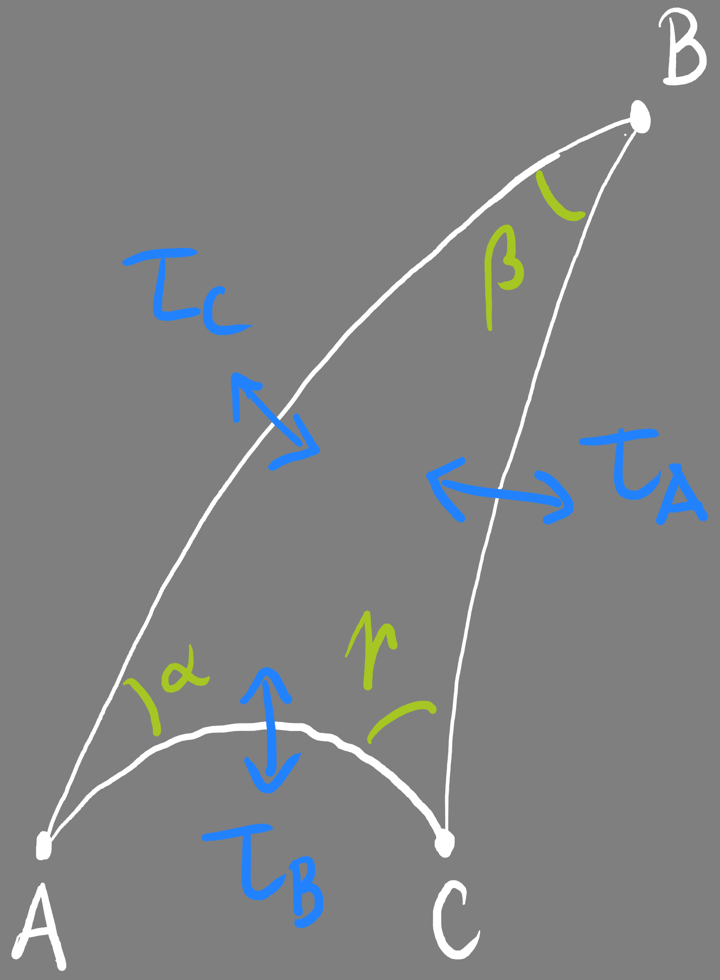
Rep_α^{DT}



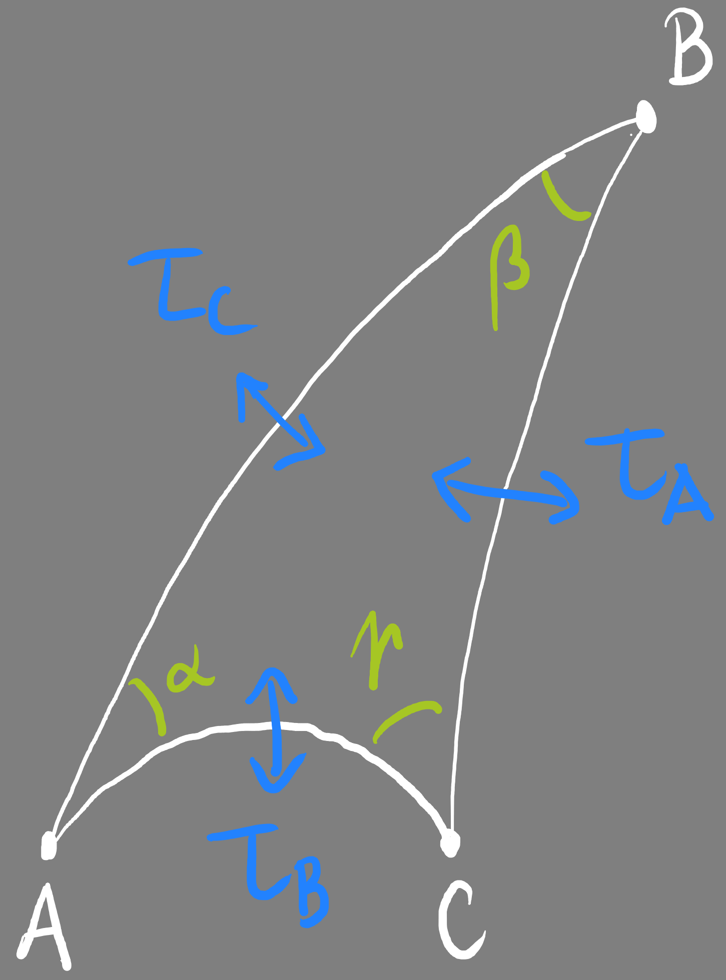


$\pi - \frac{\alpha_2}{2}$
$\Sigma = \pi$
$\pi - \frac{\alpha_3}{2}$
$\pi - \frac{\alpha_1}{2}$
$\Sigma = \pi$
$\Sigma = \pi$
$\pi - \frac{\alpha_4}{2}$
...
$\pi - \frac{\alpha_5}{2}$


B
β
α
γ
A
C

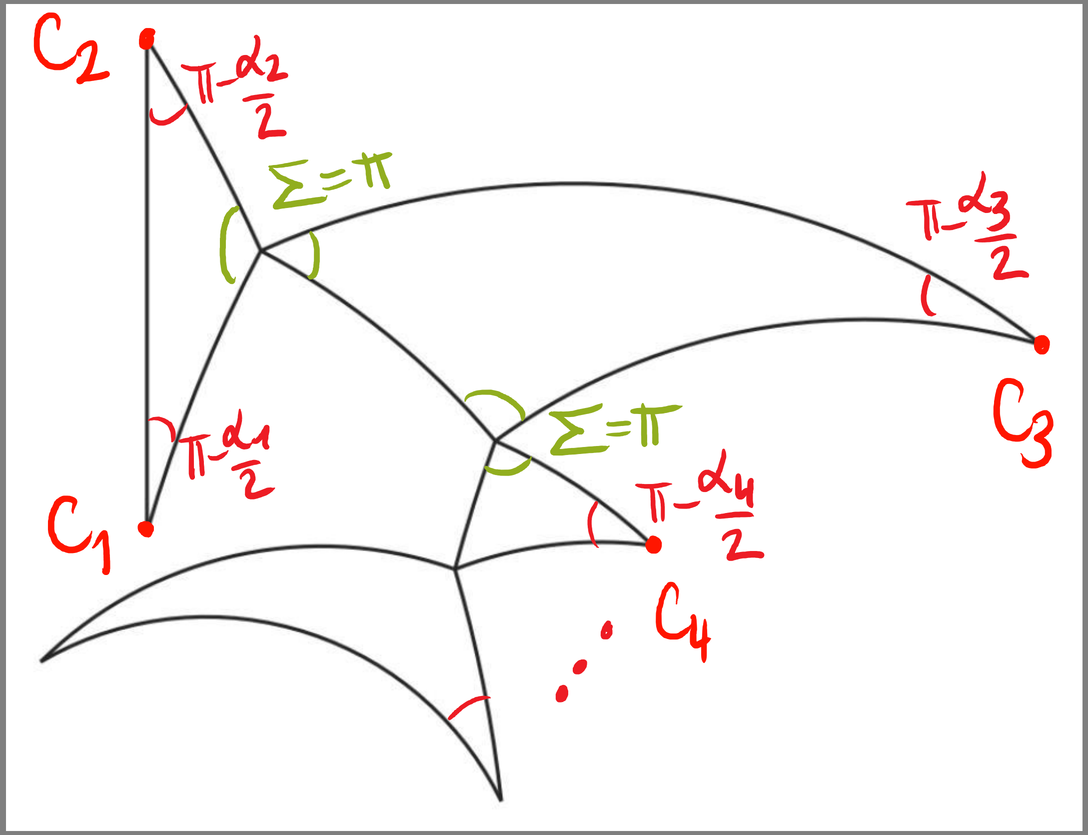


$\tau_C \tau_A =$ rotation of angle 2β around B



$\tau_C \tau_A =$ rotation of angle 2β around B

$$(\tau_C \tau_A) \cdot (\tau_A \tau_B) \cdot (\tau_B \tau_C) = 1$$

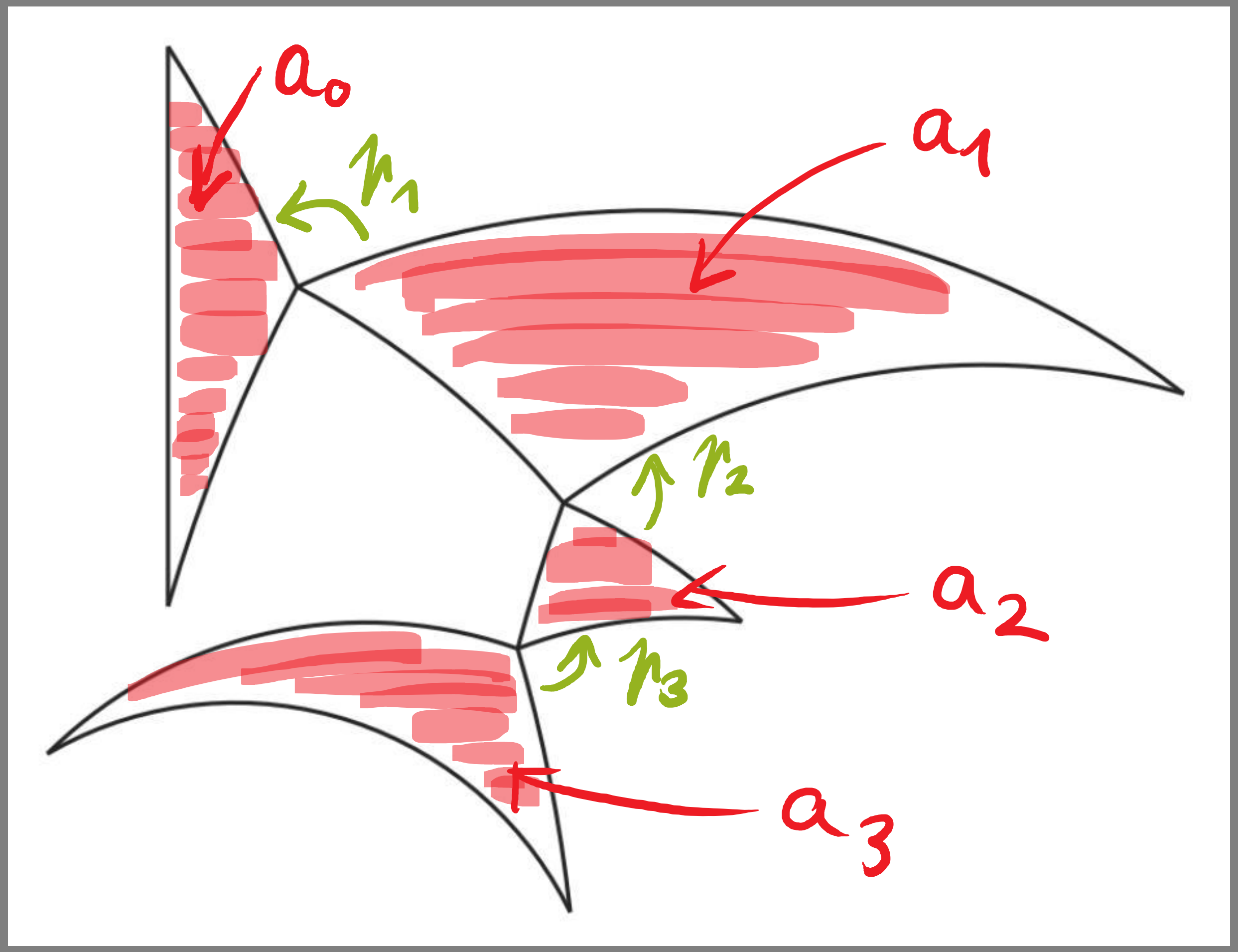


$\dashrightarrow \mathrm{Rep}_{\alpha}^{DT}$

$$\langle c_1, \dots, c_n \mid \prod_{i=1}^{n} c_i = 1 \rangle \to PSL_2\mathbb{R}$$

$$c_i \longmapsto \mathrm{rot}_{\alpha_i}(C_i)$$

$$\begin{array}{c} \mathrm{Rep}_\alpha^{DT} \\ \downarrow \\ \mathbb{CP}^{n-3} \end{array}$$



$(n=6)$

$$\sigma_i = \gamma_1 + \ldots + \gamma_i$$

$$\left[\sqrt{a_0} : e^{i\sigma_1}\sqrt{a_1} : \ldots : e^{i\sigma_{n-3}}\sqrt{a_{n-3}}\right]$$

THM (M.22')

$$\mathrm{Rep}_\alpha^{DT} \longrightarrow \mathbb{CP}^{n-3}$$

is an isomorphism of symplectic manifolds

and

$$\omega_{Goldman} = \frac{1}{2}\sum_{i=1}^{n-3} da_i \wedge d\sigma_i$$

and now...

(1) quantum representations from

$$\mathrm{Mod}(\Sigma_n) \hookrightarrow \mathrm{Rep}_\alpha^{DT}$$

(2) generalization to $G = SU(p,q)$, ...

⋮