ON REMARKABLE REPRESENTATIONS

$\pi_1\Big($  $\Big) \longrightarrow \mathrm{PSL}_2\mathbb{R}$

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Our plan today

(1) Character varieties

(2) Deroin-Tholozan representations

(3) A combinatorial model

(1) Character varieties

closed surface

Lie group

(eg: $G = PSL_2\mathbb{R}$)

$$\mathrm{Hom}(\pi_1(\Sigma_g), G)$$

closed
surface

lie group
(eg: $G = PSL_2\mathbb{R}$)

$$\mathrm{Hom}(\pi_1(\Sigma_g), G)$$

algebraic variety
inside G^{2g}

closed surface

Lie group

(eg: $G = PSL_2\mathbb{R}$)

$$\mathrm{Hom}(\pi_1(\Sigma_g), G) \circlearrowleft \mathrm{Inn}(G)$$

algebraic variety inside G^{2g}

$$\mathrm{Rep}(\Sigma_g, G) := \mathrm{Hom}(\pi_1(\Sigma_g), G) / \mathrm{Inn}(G)$$

character variety of (Σ_g, G)

punctured surface

lie group

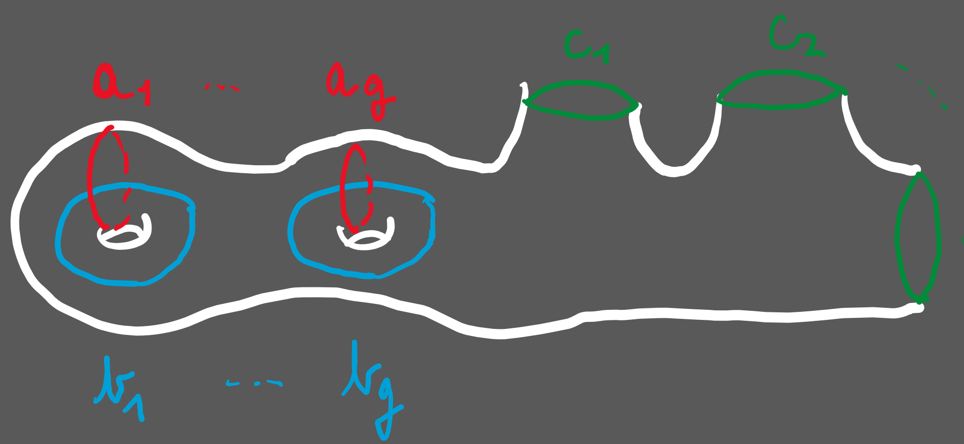
$$\mathrm{Hom}(\pi_1(\Sigma_{g,n}), G)$$

punctured surface

lie group

$$\mathrm{Hom}\left(\pi_1(\Sigma_{g,n}), G\right)$$

$\pi_1\Bigg($



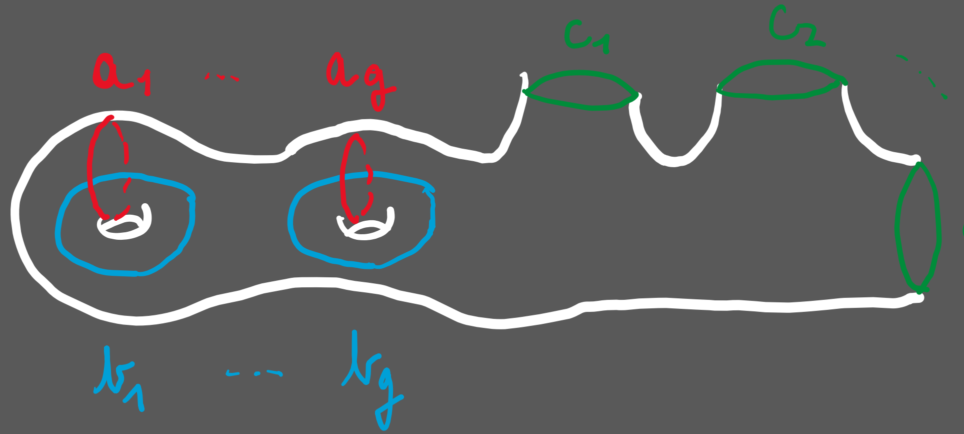
$\Bigg) = \left\langle a_i, b_i, c_j \;\middle|\; \prod_{i=1}^{g} [a_i, b_i] = \prod_{j=1}^{n} c_j \right\rangle$

punctured surface

lie group

$$\mathrm{Hom}(\pi_1(\Sigma_{g,n}), G)$$

$\pi_1\Big($

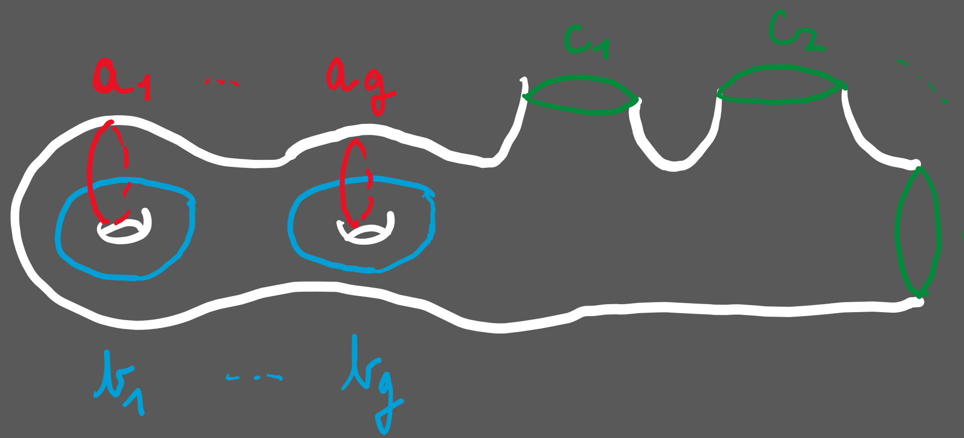


$\Big) \cong F_{2g+n-1}$

punctured surface

lie group

$$\mathrm{Hom}(\pi_1(\Sigma_{g,n}), G) \cong G^{2g+n-1}$$

$$\pi_1\Bigg($$  $$\Bigg) \cong F_{2g+n-1}$$

punctured surface

lie group

$$\mathrm{Hom}_{\mathcal{C}}(\pi_1(\Sigma_{g,n}), G)$$

collection of n conjugacy classes $C_1, \dots, C_n \in G/_{\mathrm{conj}}$

$\phi(c_i) \in C_i$

punctured surface

lie group

$$\mathrm{Hom}_{\mathcal{C}}\left(\pi_1(\Sigma_{g,n}),\ G\right) \circlearrowleft \mathrm{Inn}(G)$$

collection of n conjugacy classes $C_1, \dots, C_n \in G/_{\mathrm{conj}}$

$\phi(c_i) \in C_i$

$$\mathrm{Rep}_{\mathcal{C}}(\Sigma_{g,n}, G) := \mathrm{Hom}_{\mathcal{C}}(\pi_1(\Sigma_{g,n}), G) / \mathrm{Inn}(G)$$

relative character variety of $(\Sigma_{g,n}, G)$

main task: study

the topology / geometry

of (relative) character varieties

$$\mathrm{Rep}_{\mathcal{C}}(\Sigma_{g,n}, G)$$

main task: study

the topology / geometry

of (relative) character varieties

$\mathrm{Rep}_{\mathcal{C}}(\Sigma_{g,n}, G)$

(1) Goldman symplectic structure

main task: study
the topology/geometry
of (relative) character varieties

$$\mathrm{Rep}_{\mathcal{C}}(\Sigma_{g,n}, G)$$

(1) Goldman symplectic structure

(2) Toledo number

even dimensional manifold

closed & non-degenerate 2-form

(M, ω)

symplectic manifold

even dimensional manifold

(M, ω)

closed & non-degenerate 2-form

symplectic manifold

section of T^*M $\overset{\omega}{\longleftrightarrow}$ section of TM

even dimensional manifold

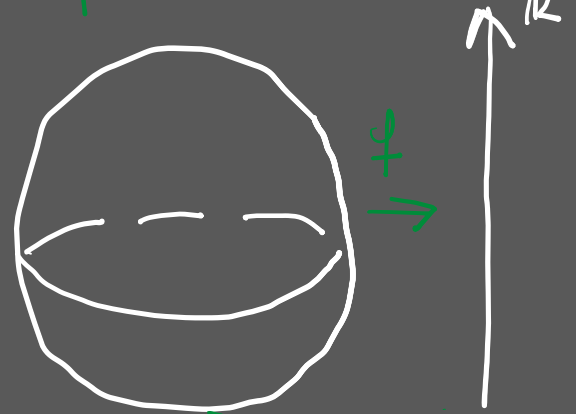
closed & non-degenerate 2-form

(M, ω)

symplectic manifold

smooth function $\rightsquigarrow$ section of T^*M $\overset{\omega}{\longleftrightarrow}$ section of TM

$f : M \to \mathbb{R}$ $\quad\quad$ df



$$(M, \omega)$$

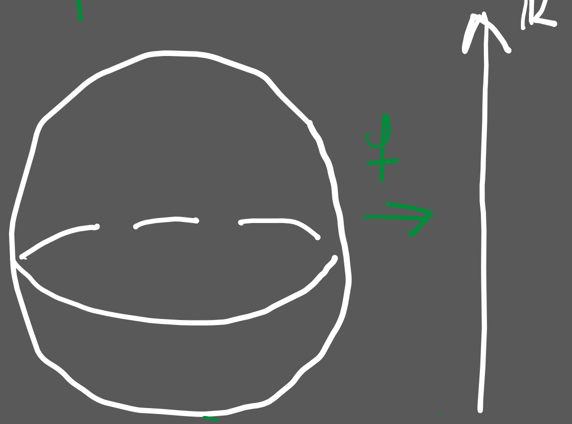
even dimensional manifold

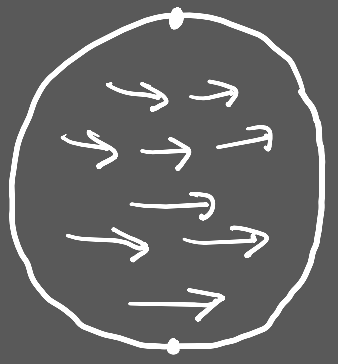
closed & non-degenerate 2-form

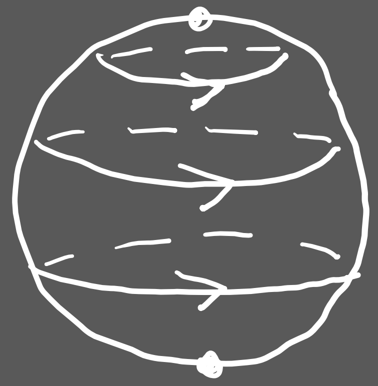
symplectic manifold

smooth function $\rightsquigarrow$ section of T^*M $\overset{\omega}{\longleftrightarrow}$ section of TM $\rightsquigarrow$ flow

$f: M \to \mathbb{R}$ df X_f $\Phi_f^t : M \to M,\ t \in \mathbb{R}$







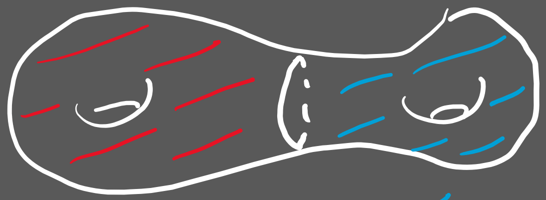
$$\mathrm{Tol} : \mathrm{Rep}(\Sigma_{g,n}, G) \longrightarrow \mathbb{R}$$

Toledo number

[Burger–Iozzi–Wienhard]

* continuous
* locally constant on <u>relative</u> character variety
* (additivity)

$$\mathrm{Tol}(\phi) = \mathrm{Tol}(\phi|_{\pi_1(\Sigma_1)}) + \mathrm{Tol}(\phi|_{\pi_1(\Sigma_2)})$$



Σ_1 Σ_2

* (Milnor–Wood inequality)

$$|\mathrm{Tol}| \le |\chi(\Sigma_{g,n})| \cdot \mathrm{rank}(G)$$

* (...)

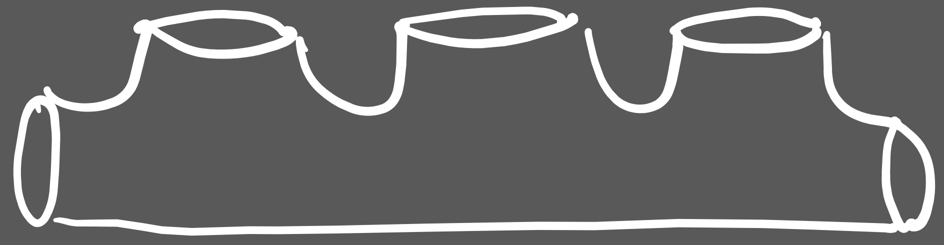
(relative) character variety = symplectic manifold associated to $(\Sigma_{g,n}, G)$

$$\left(\mathrm{Rep}_{\mathcal{C}}(\Sigma_{g,n}, G) \,,\; \omega_{\mathrm{Goldman}} \right)$$

(2) Deroin - Tholozan representations

A remarkable example : Deroin-Tholozan representations

surface = $\Sigma_{0,n}$



Lie group = $PSL_2\mathbb{R}$

compact component

$$\mathrm{Rep}^{DT}_{\alpha}(\Sigma_{0,n}, PSL_2\mathbb{R}) \subseteq \mathrm{Rep}_{\alpha}(\Sigma_{0,n}, PSL_2\mathbb{R})$$

elliptic conjugacy classes

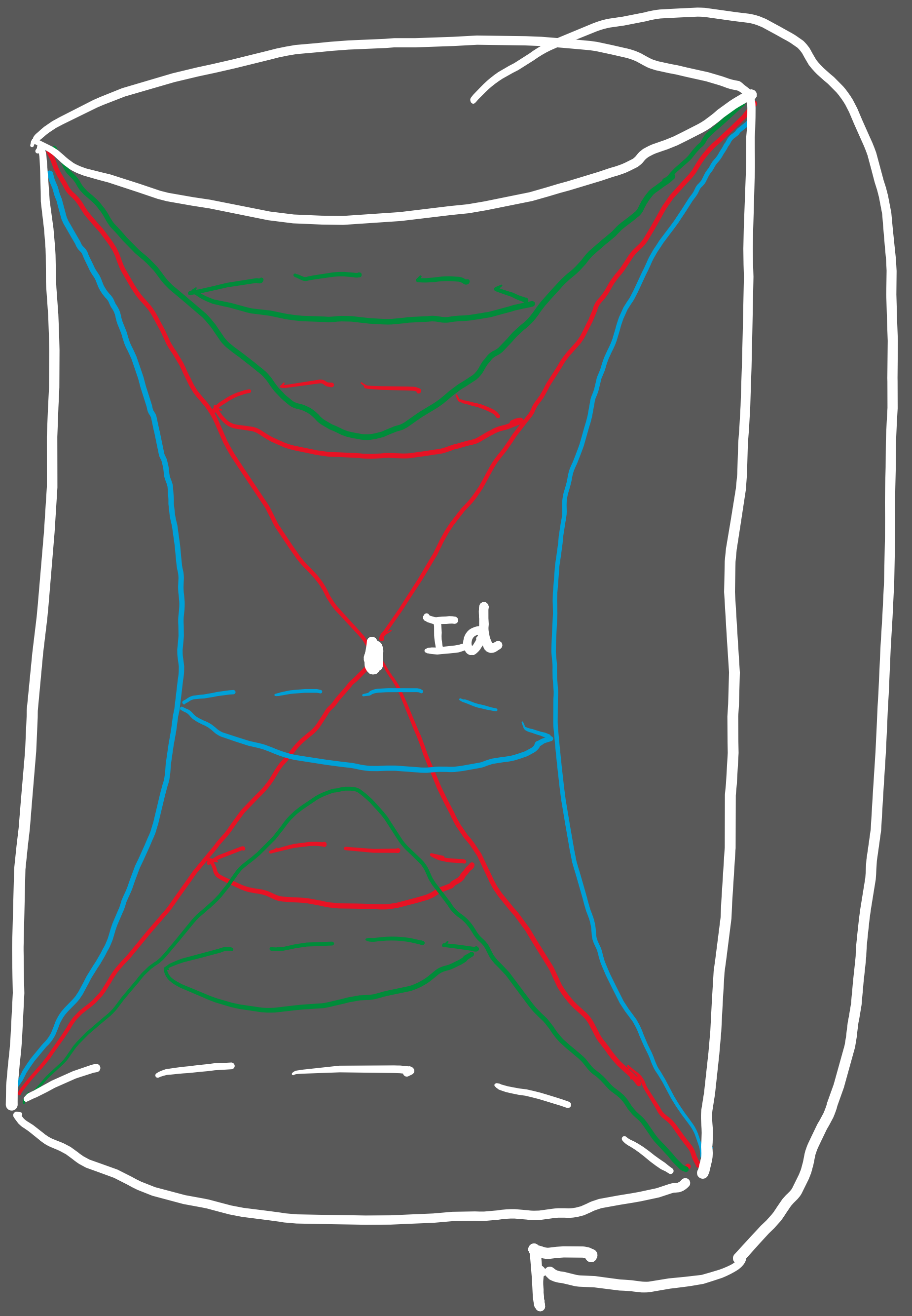
$PSL_2\mathbb{R}$

elliptic $= \left\{ \begin{array}{c} \text{unique fixed} \\ \text{point in } \mathbb{H}^2 \end{array} \right\}$

$\cong \mathbb{H}^2 \times (0, 2\pi)$

fixed point

ϑ = angle of rotation



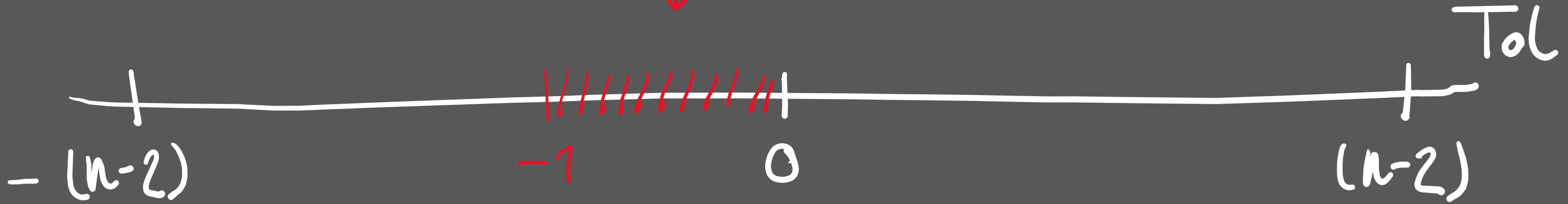
$$\alpha = (\alpha_1, \dots, \alpha_n) \in (0, 2\pi)^n$$

crucial angles condition

$$2\pi \cdot n > \alpha_1 + \dots + \alpha_n > 2\pi \cdot (n-1)$$

$$\mathrm{Rep}^{DT}_{\alpha}(\Sigma_{0,n}, PSL_2\mathbb{R}) \subseteq \mathrm{Rep}_{\alpha}(\Sigma_{0,n}, PSL_2\mathbb{R})$$

$\mathrm{Tol} \in (-1, 0)$



THM [Deroin – Tholozan]

$$\left(\mathrm{Rep}^{DT}_{\alpha}\left(\Sigma_{0,n}, \mathrm{PSL}_2\mathbb{R}\right), \omega_{\mathrm{Goldman}}\right) \cong \left(\mathbb{CP}^{n-3}, \lambda \cdot \omega_{FS}\right)$$

$$\lambda = \alpha_1 + \dots + \alpha_n - 2\pi(n-1)$$
$$= -\frac{1}{2\pi}\mathrm{Tol}(\phi)$$

THM [Deroin – Tholozan]

$$\left(\mathrm{Rep}^{\mathrm{DT}}_{\alpha}\left(\Sigma_{0,n}, \mathrm{PSL}_2\mathbb{R}\right), \omega_{\mathrm{Goldman}}\right) \cong \left(\mathbb{CP}^{n-3}, \lambda\cdot\omega_{FS}\right)$$

Compact

$$\lambda = \alpha_1 + \dots + \alpha_n - 2\pi(n-1)$$
$$= -\frac{1}{2\pi}\mathrm{Tol}(\phi)$$

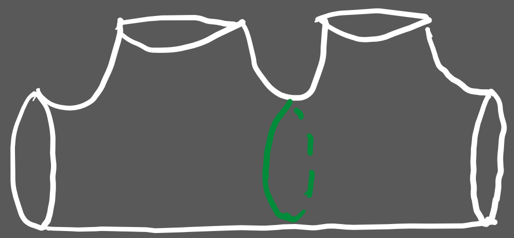
THM [Deroin–Tholozan]

$$\left(\mathrm{Rep}^{\mathrm{DT}}_{\alpha}(\Sigma_{0,n}, PSL_2\mathbb{R}), \omega_{\mathrm{Goldman}}\right) \cong \left(\mathbb{CP}^{n-3}, \lambda \cdot \omega_{FS}\right)$$

compact

totally elliptic

$$\lambda = \alpha_1 + \dots + \alpha_n - 2\pi(n-1) = -\frac{1}{2\pi}\mathrm{Tol}(\phi)$$

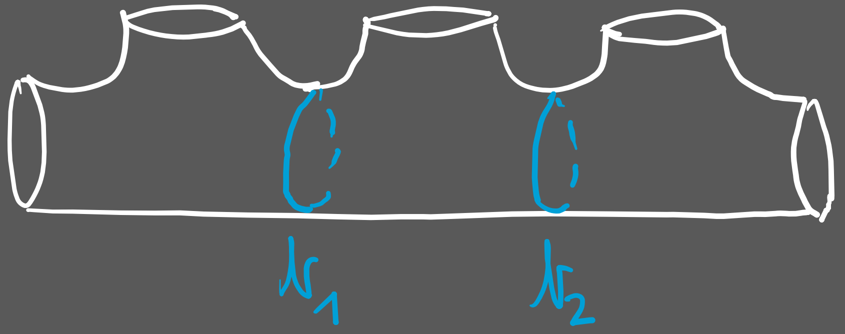


$\mapsto$ elliptic

proof: $\quad T^{n-3} \circlearrowright \left(\mathrm{Rep}_{\alpha}^{DT} , \omega_{\mathrm{Goldman}} \right)$

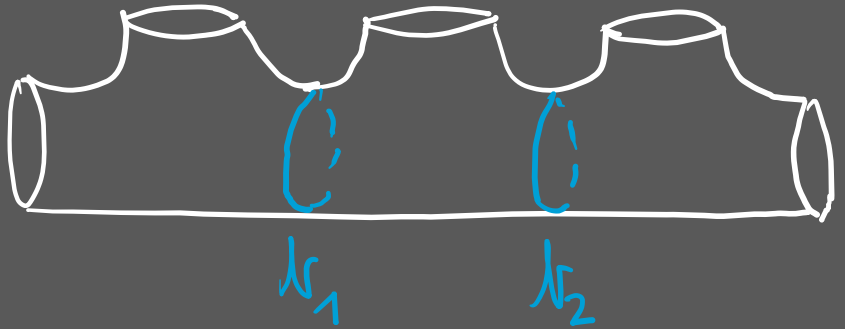
<u>proof</u>: $T^{n-3} \circlearrowright (\mathrm{Rep}_{\alpha}^{DT}, \omega_{\mathrm{Goldman}})$

$(n=5)$



proof: $T^{n-3} \circlearrowright (\mathrm{Rep}_\alpha^{OT}, \omega_{Goldman})$

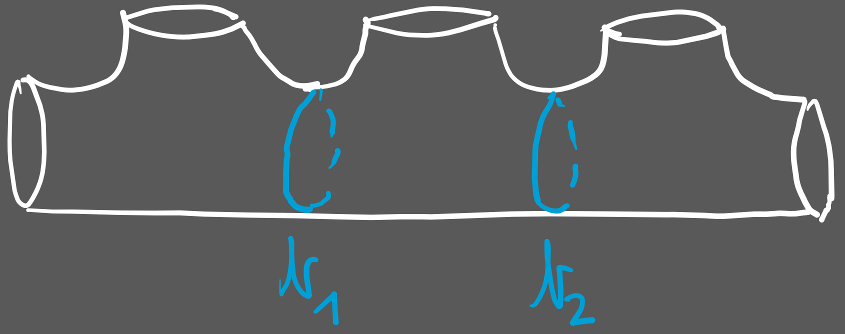
(n=5)



$\rightsquigarrow \quad \vartheta_{\gamma_i} : \mathrm{Rep}_\alpha^{OT} \longrightarrow (0, 2\pi)$

proof: $T^{n-3} \circlearrowright \left(\mathrm{Rep}_{\alpha}^{DT}, \omega_{\mathrm{Goldman}} \right)$

$(n=5)$



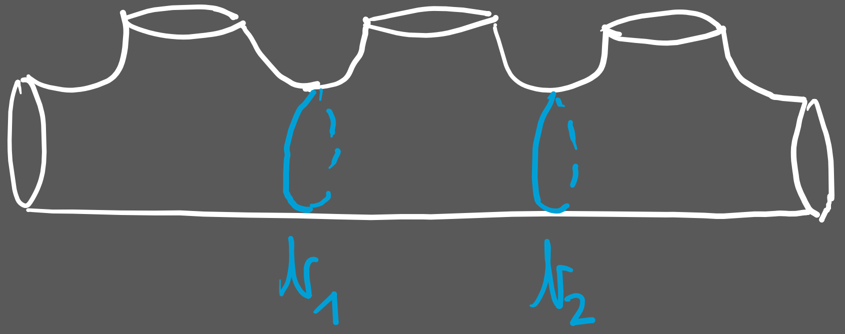
$\rightsquigarrow \vartheta_{\gamma_i} : \mathrm{Rep}_{\alpha}^{DT} \longrightarrow (0, 2\pi)$

$\rightsquigarrow$ periodic & commutative

flows $\Phi_{\gamma_i} : \mathbb{R} \times \mathrm{Rep}_{\alpha}^{DT} \to \mathrm{Rep}_{\alpha}^{DT}$

proof: $T^{n-3} \curvearrowright \left(\mathrm{Rep}_\alpha^{OT}, \omega_{\mathrm{Goldman}}\right)$

$(n=5)$



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$= T^{n-3} \curvearrowright \left(\mathrm{Rep}_\alpha^{OT}, \omega_{\mathrm{Goldman}}\right)$

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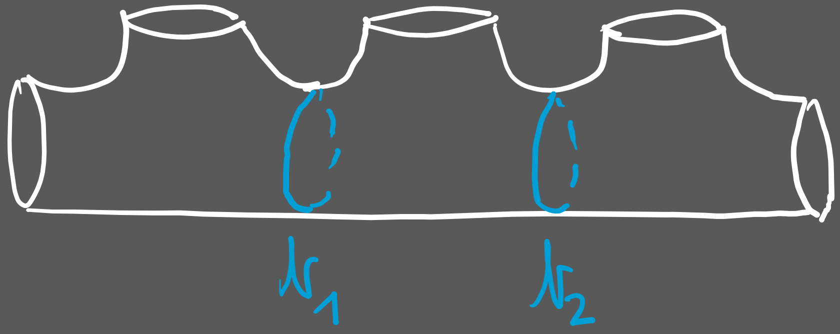
$(n=5)$



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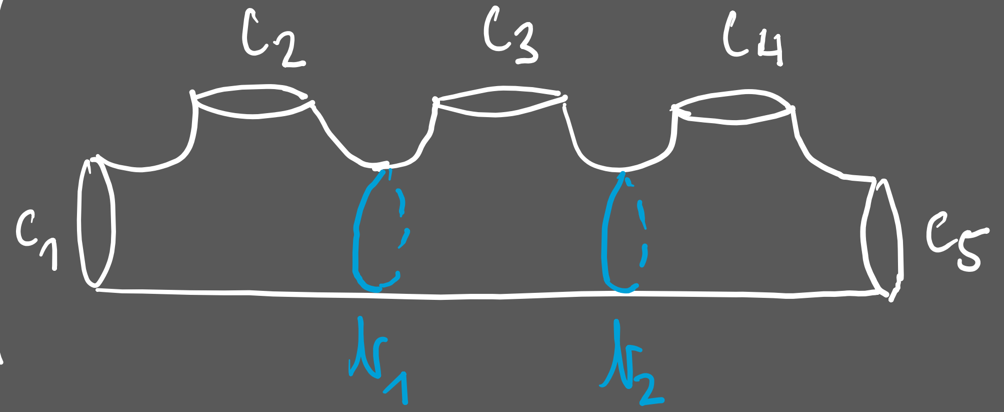
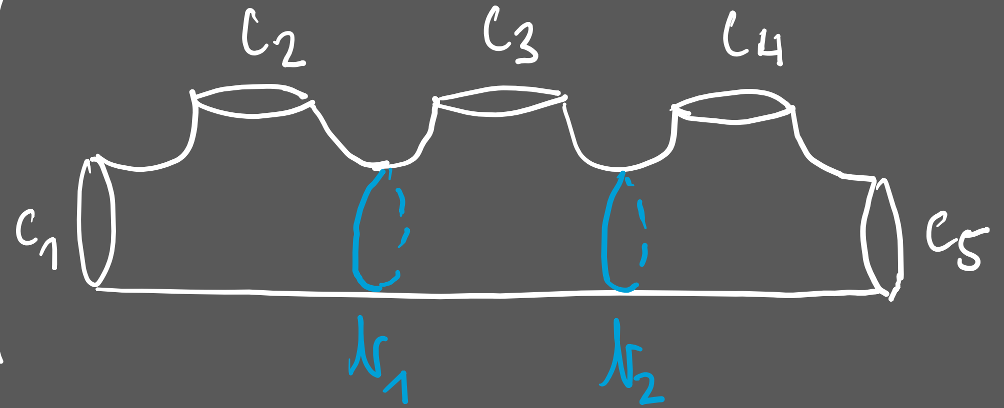
$= T^{n-3} \curvearrowright \left(\mathrm{Rep}_\alpha^{\mathrm{OT}}, \omega_{\mathrm{Goldman}}\right)$

Delzant's classification $\Longrightarrow$ $\mathrm{Rep}_\alpha^{\mathrm{OT}} \cong \mathbb{CP}^{n-3}$ $\square$

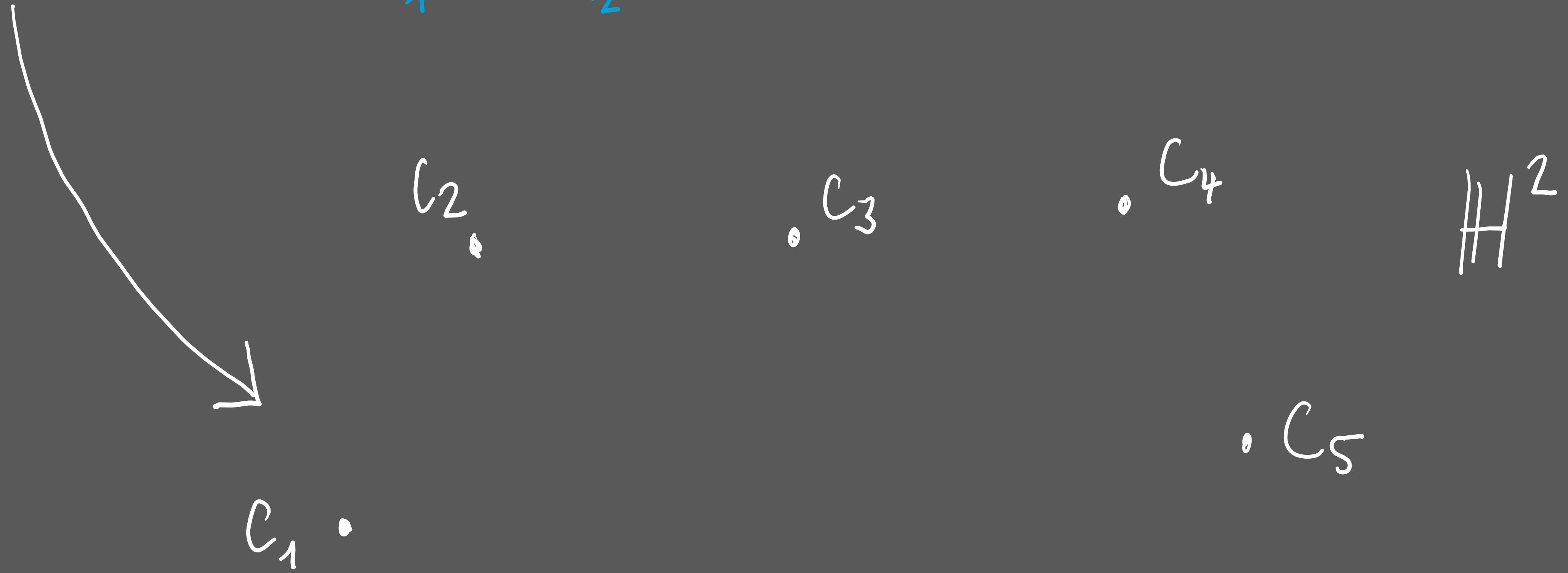
(3) A combinatorial model



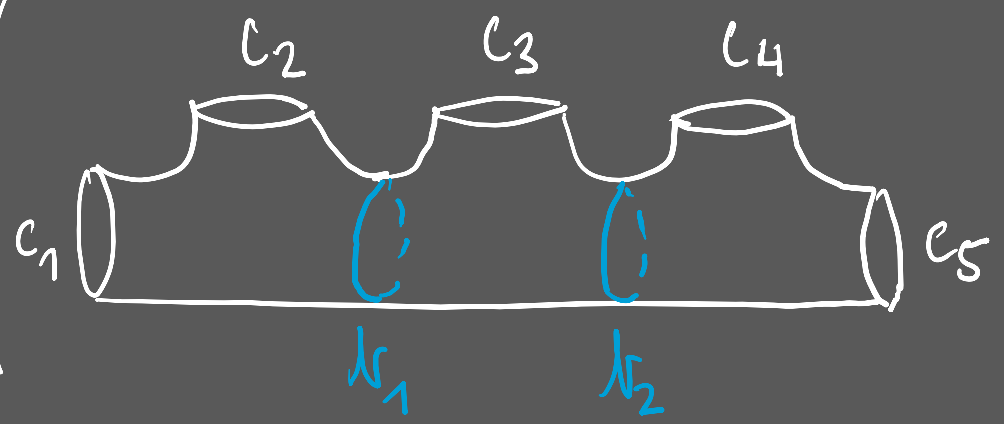
$$\pi_1\left(\quad \right) \xrightarrow{\phi} PSL_2\mathbb{R}, \qquad \vartheta(\phi(c_i)) = \alpha_i$$

$\pi_1\Big($ 

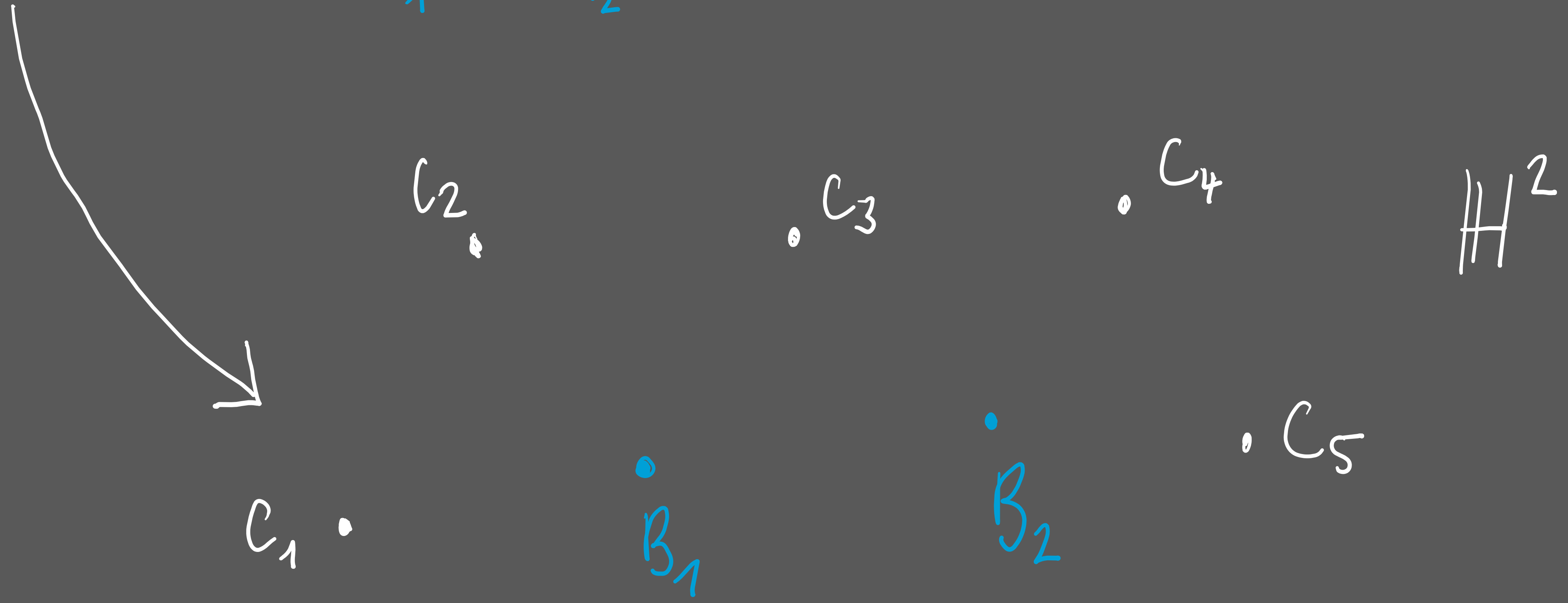
$\Big) \xrightarrow{\phi} PSL_2\mathbb{R}$, $\vartheta(\phi(c_i)) = \alpha_i$

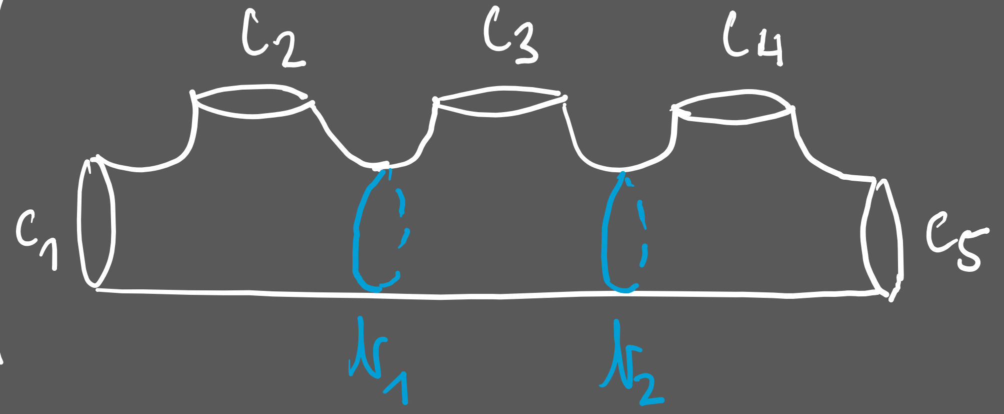


$\pi_1 \Big($

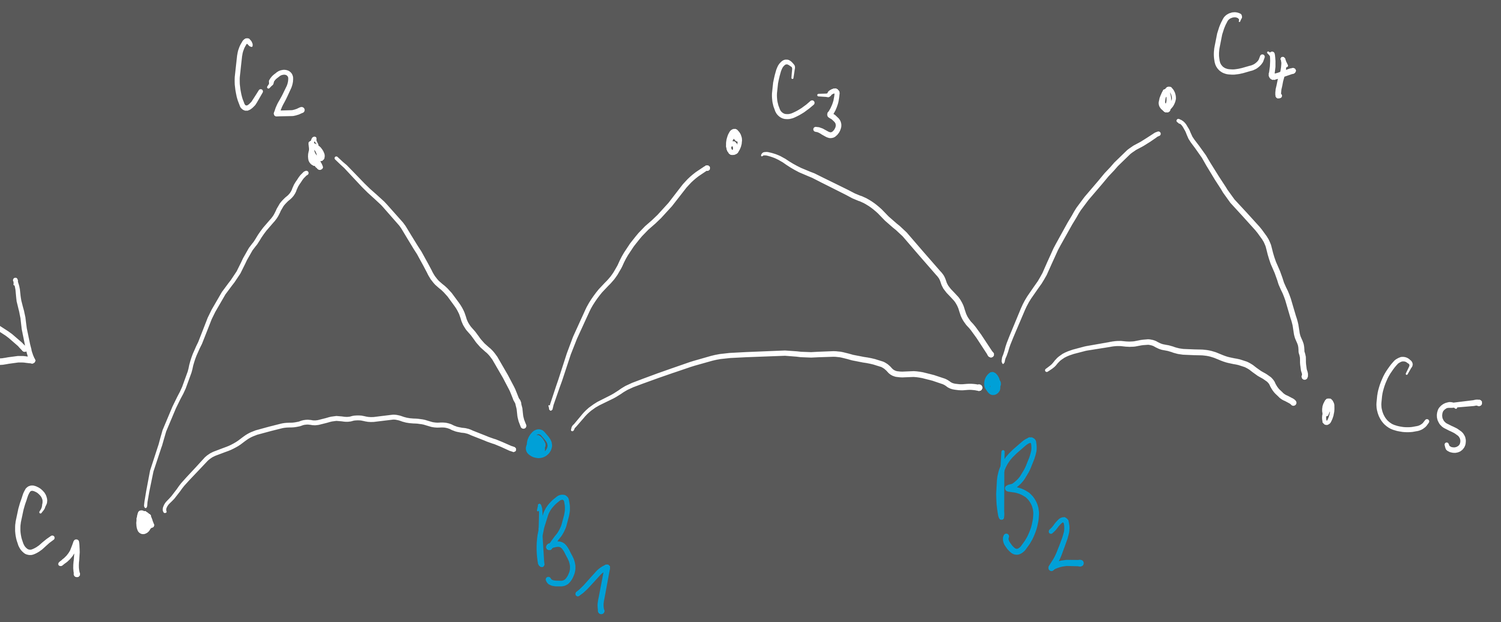


$\Big) \xrightarrow{\phi} PSL_2\mathbb{R}$, $\vartheta(\phi(c_i)) = \alpha_i$

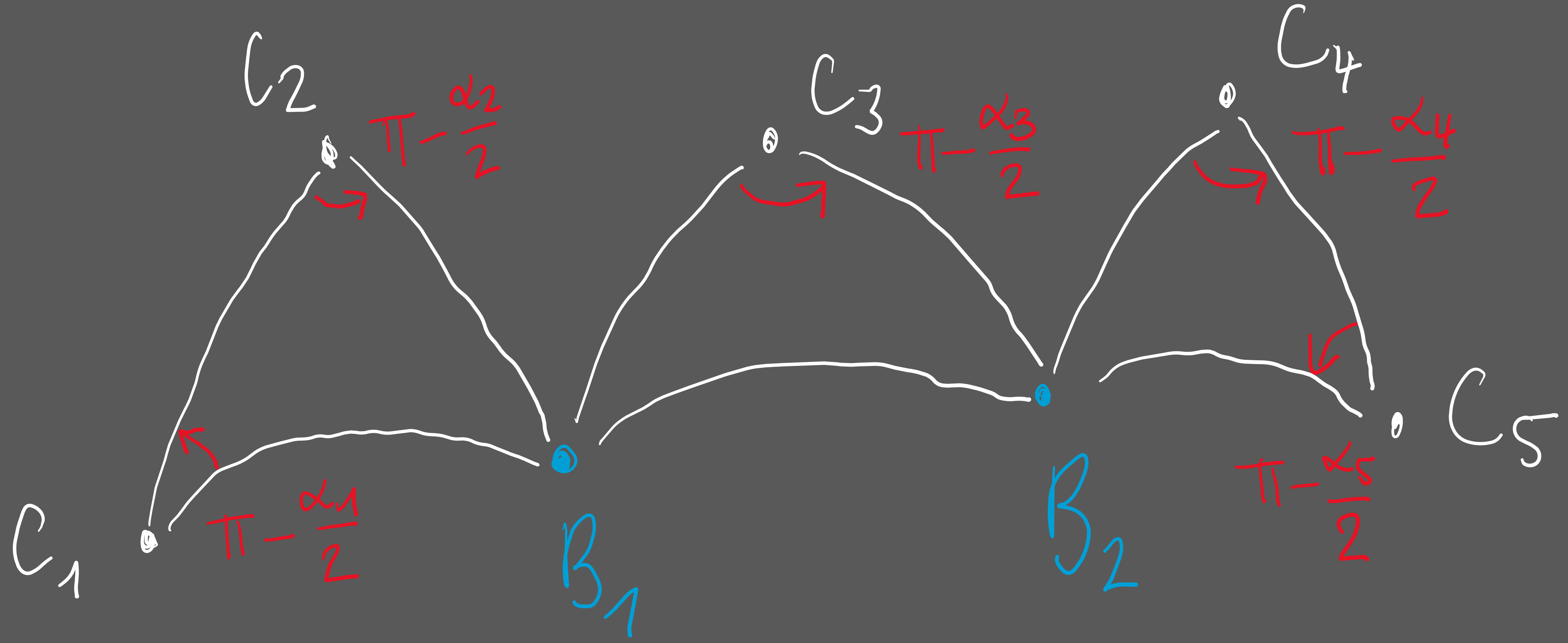


$\pi_1\Big($ 

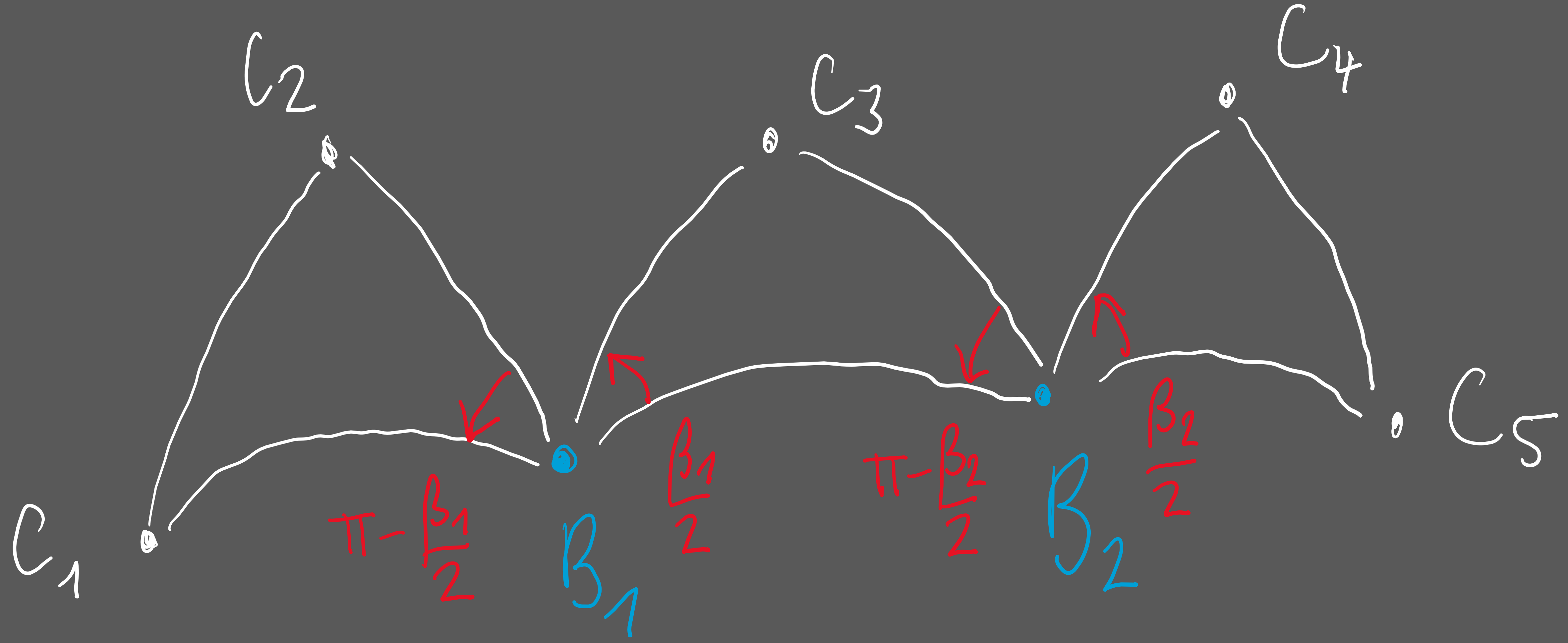
$\Big) \xrightarrow{\phi} PSL_2\mathbb{R}$, $\vartheta(\phi(c_i)) = \alpha_i$



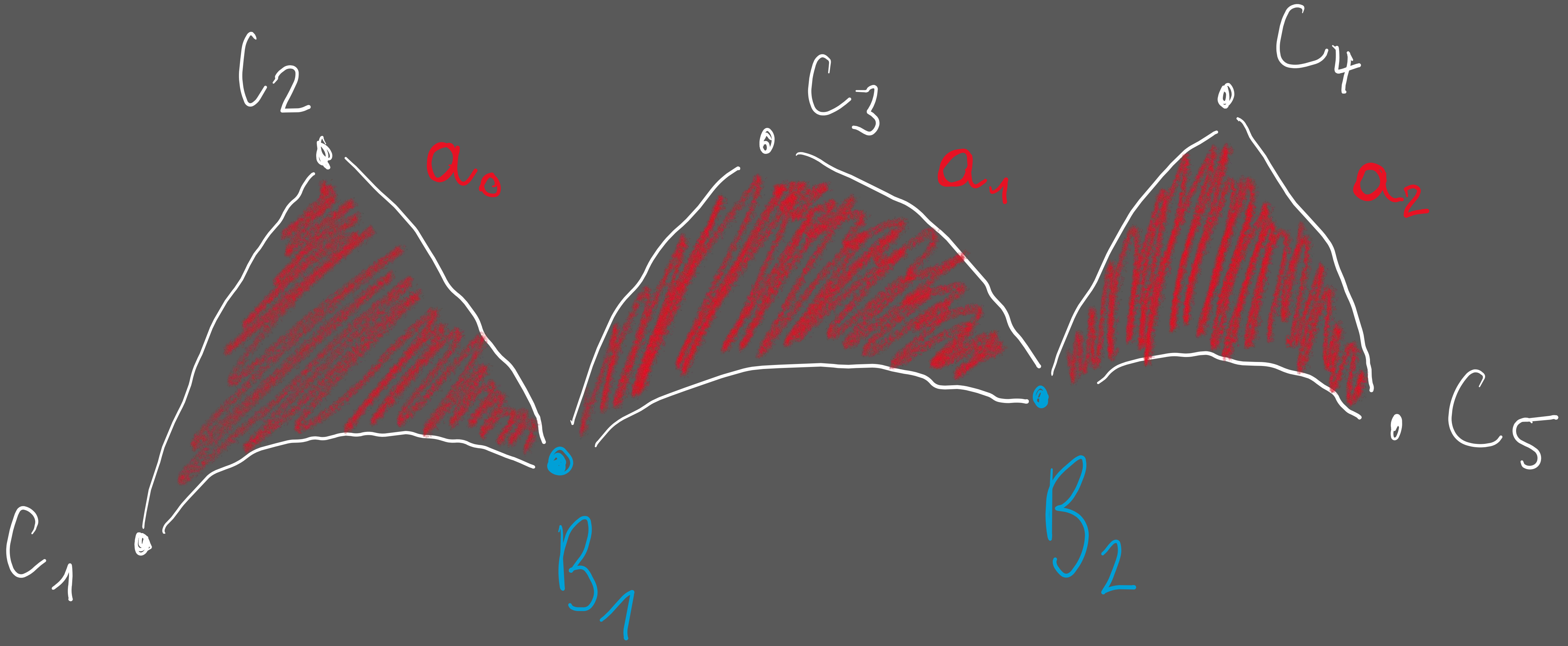
$\mathbb{H}^2$



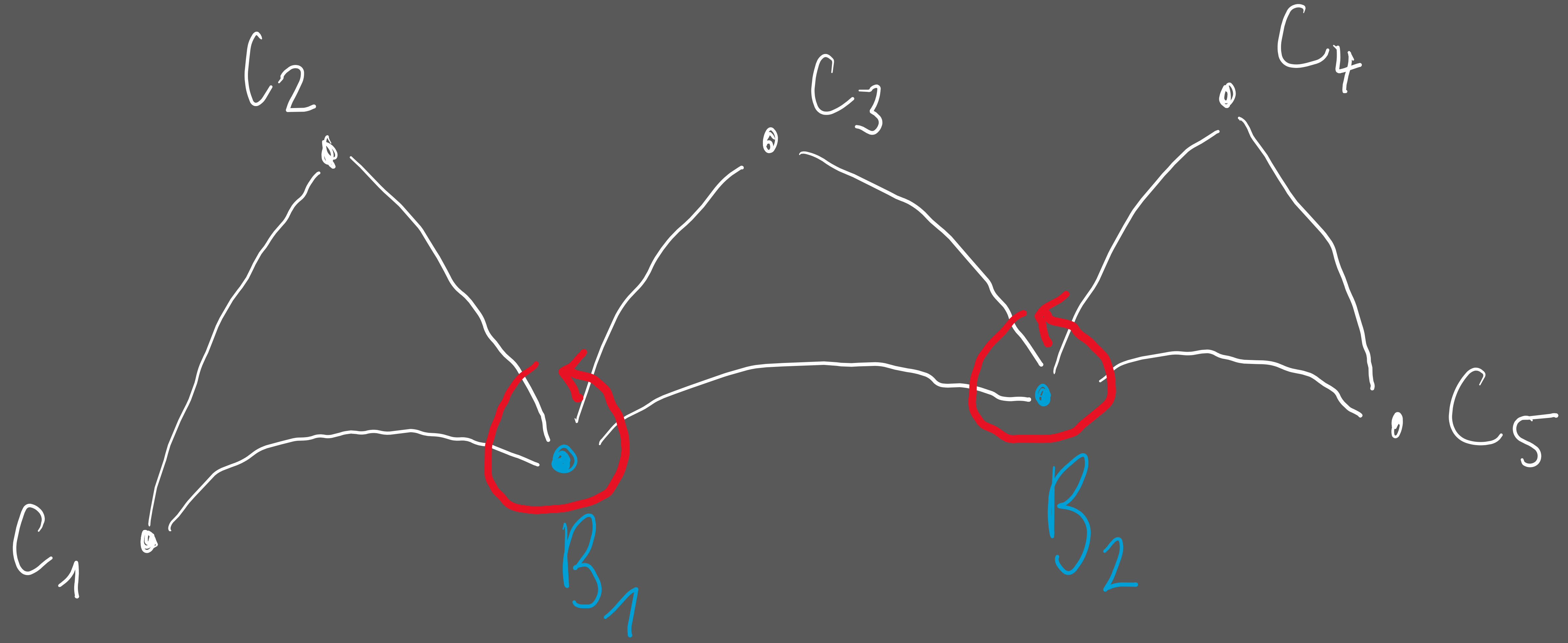
C_1
C_2
C_3
C_4
C_5
B_1
B_2
$\pi - \frac{\alpha_1}{2}$
$\pi - \frac{\alpha_2}{2}$
$\pi - \frac{\alpha_3}{2}$
$\pi - \frac{\alpha_4}{2}$
$\pi - \frac{\alpha_5}{2}$

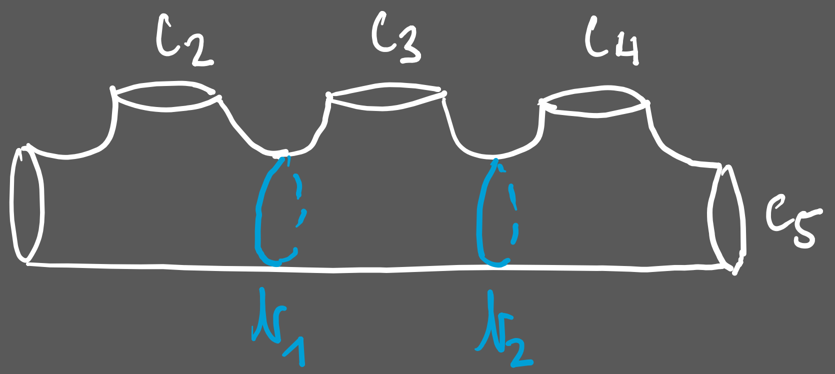


$$\beta_1 = \vartheta(\phi(I_1)) \quad , \quad \beta_2 = \vartheta(\phi(I_2))$$

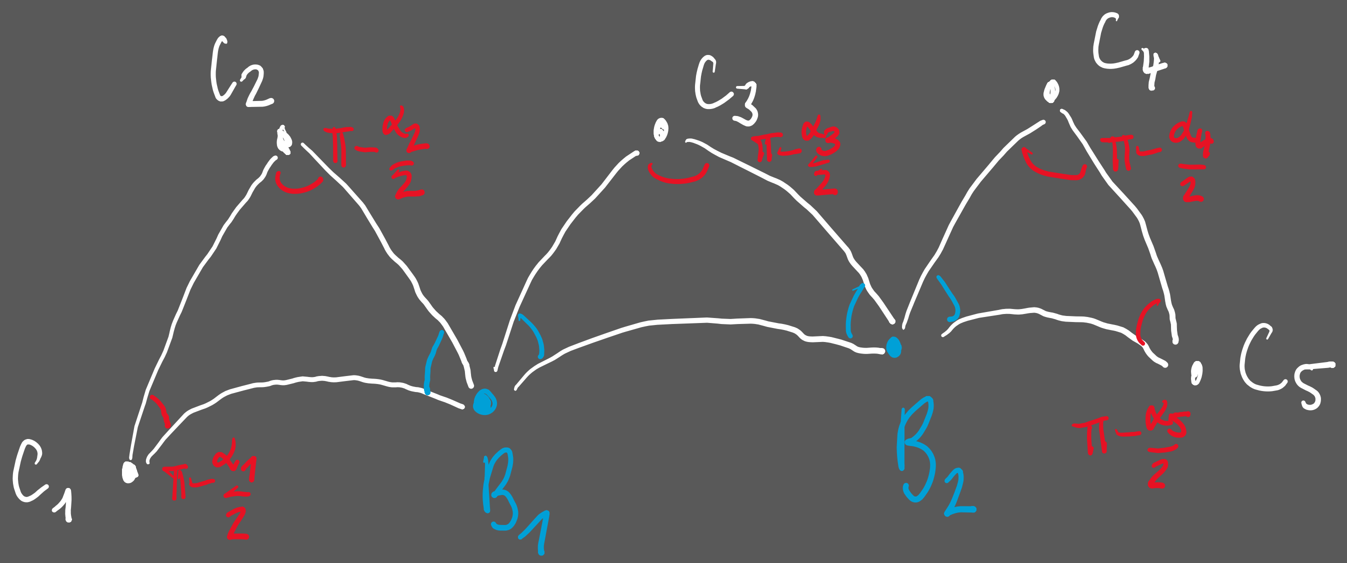


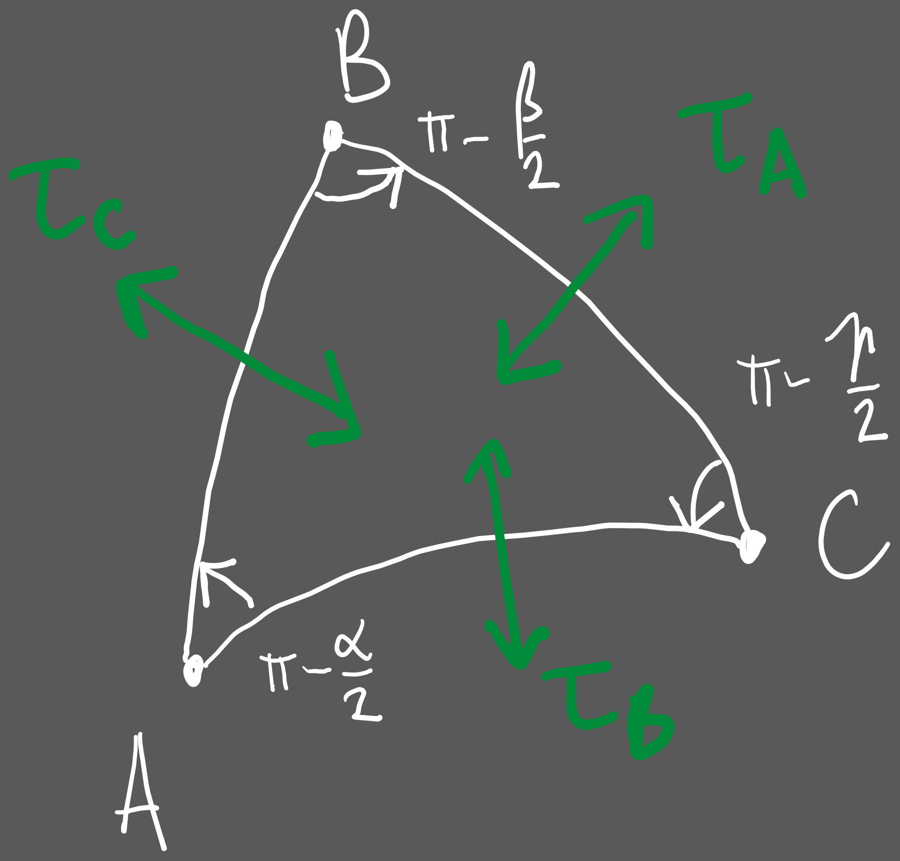
$$a_0 + a_1 + a_2 = \frac{\lambda}{2} = -\frac{1}{4\pi} \mathrm{Tol}(\phi)$$


C_1
C_2
C_3
C_4
C_5
B_1
B_2

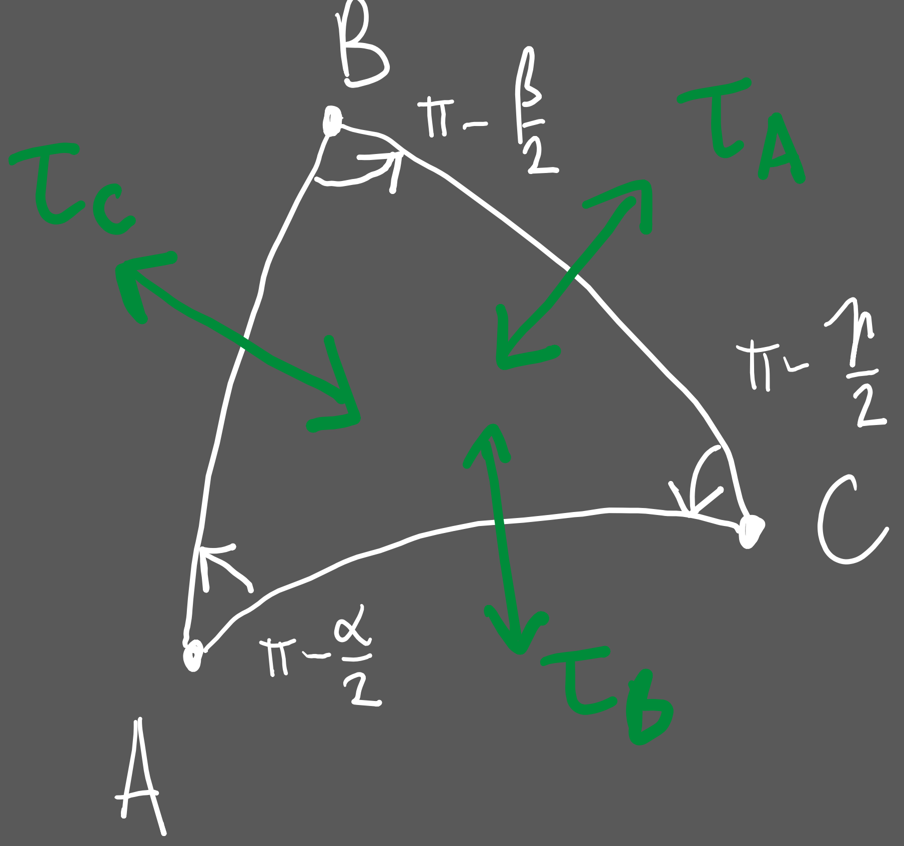
$\pi_1 \Big($  $\Big) \xrightarrow{\ \phi\ } PSL_2\mathbb{R}$

$\phi(c_i) :=$
rotation of angle
α_i around C_i

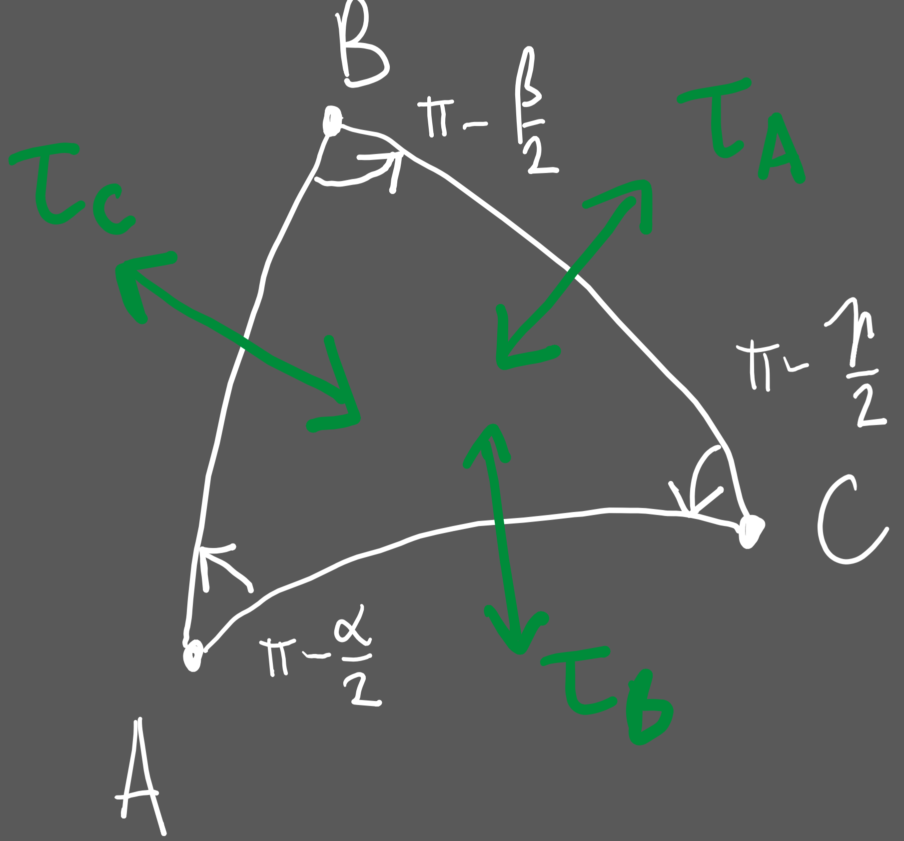




$$\tau_C \tau_A =$$

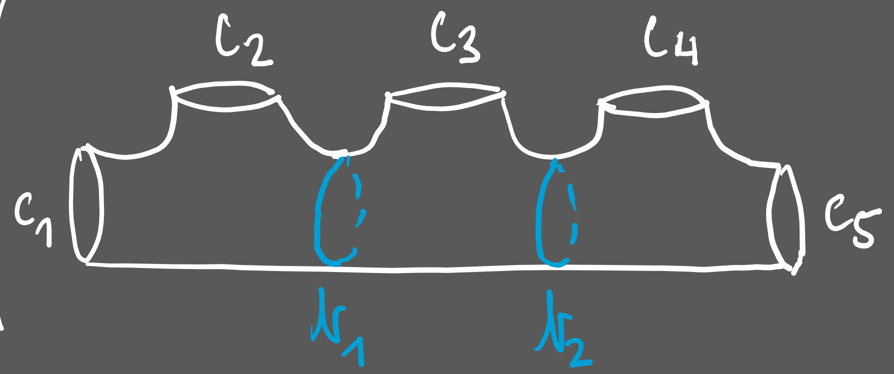


$\tau_C \tau_A =$ clockwise rotation of angle β around B

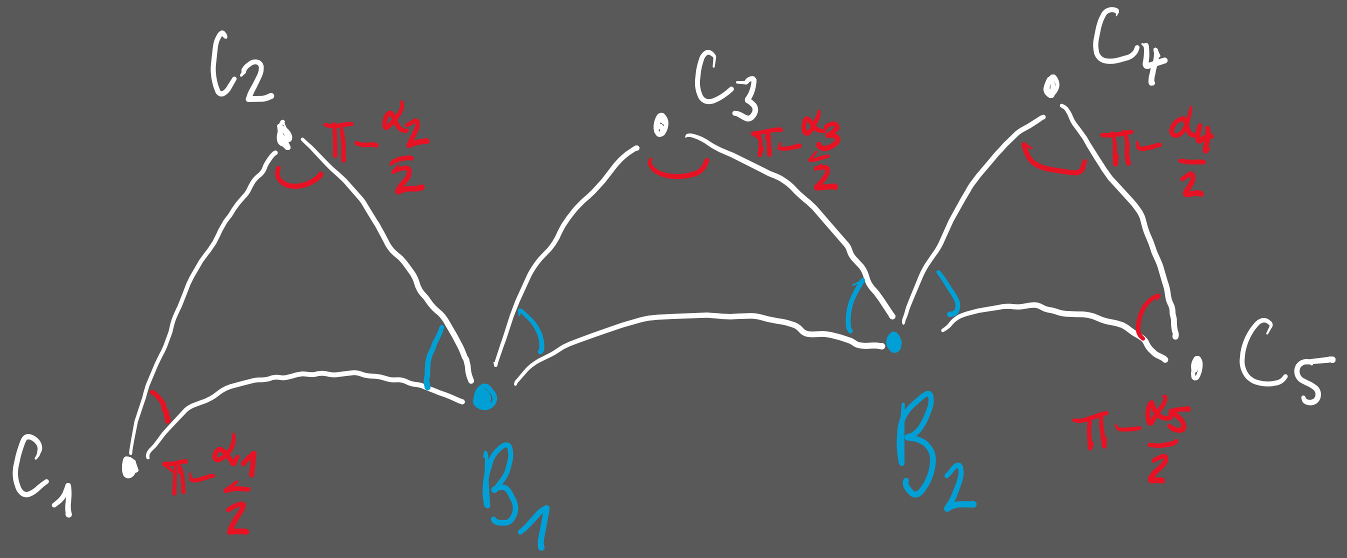


$$\tau_C \tau_A = \begin{array}{l}\text{clockwise rotation}\\ \text{of angle } \beta\\ \text{around } B\end{array}$$

$$(\tau_C \tau_A) \cdot (\tau_A \tau_B) \cdot (\tau_B \cdot \tau_C) = 1$$

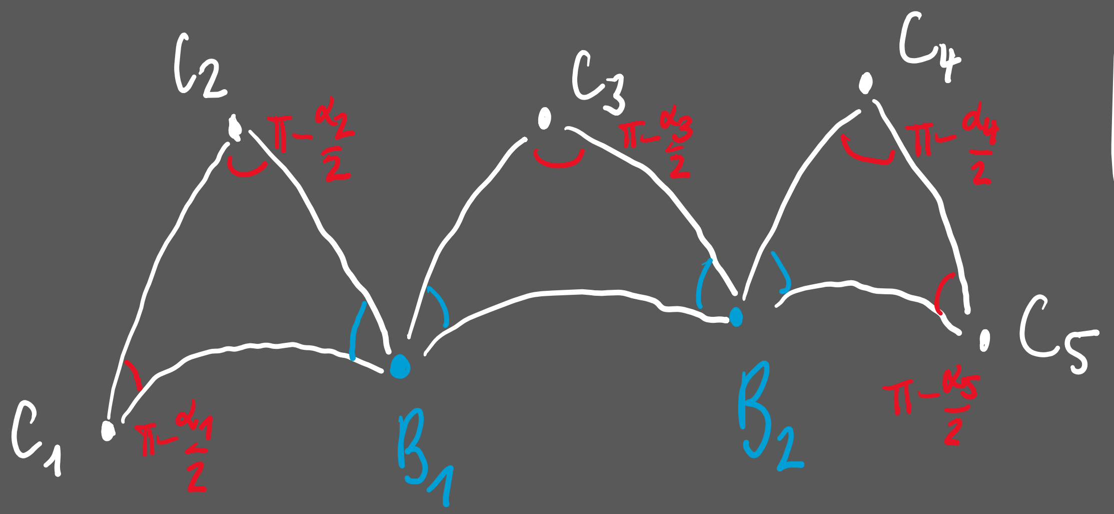
$\pi_1 \Big($  $\Big) \xrightarrow{\phi} PSL_2\mathbb{R}$

$\phi(c_i) :=$ rotation of angle α_i around C_i



THM [M.]

$$\mathrm{Rep}^{DT}_{\alpha}(\Sigma_{0,n}, PSL_2\mathbb{R}) \xleftrightarrow{1-1} \left\{ \vphantom{\int} \right.$$

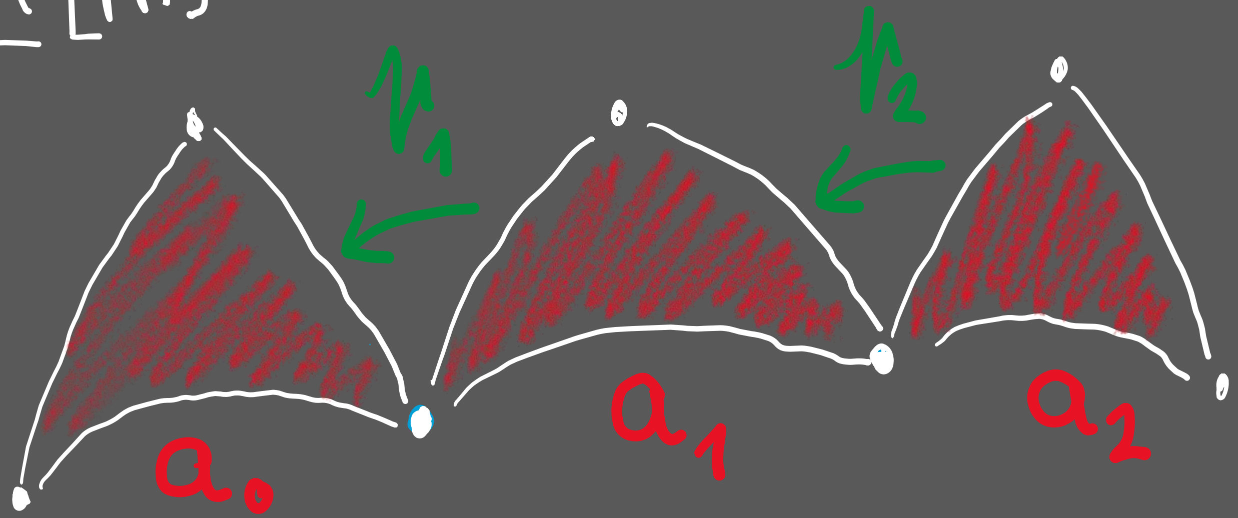


$$\left. \vphantom{\int} \right\}$$

$$\cap|$$

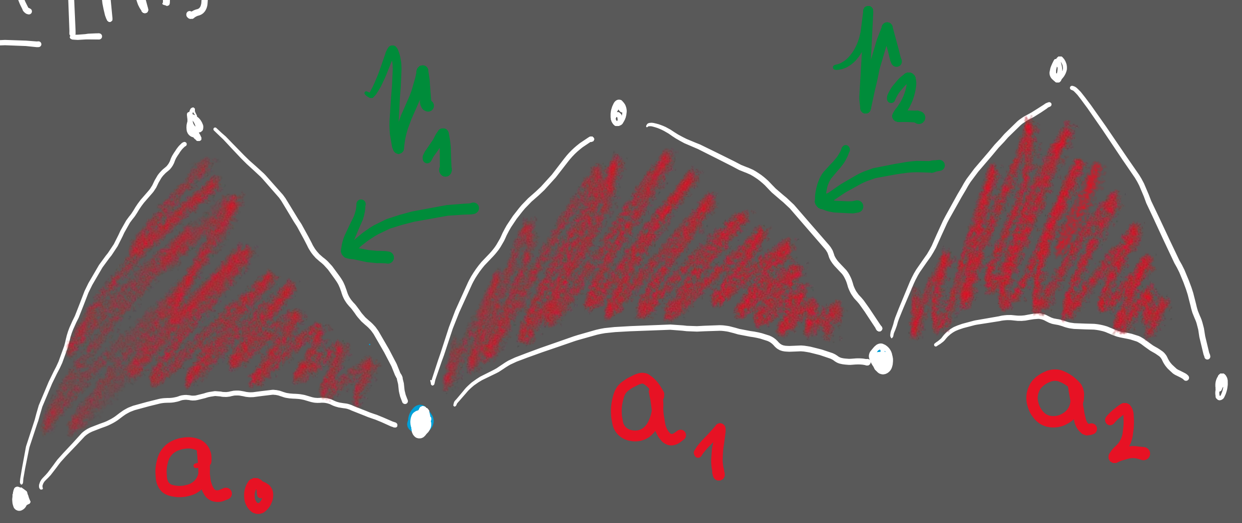
$$\mathbb{H}^n \times \mathbb{H}^{n-3} \Big/ PSL_2\mathbb{R}$$

THM [M.]



$\sigma_i := \gamma_1 + \ldots + \gamma_i$

THM [M.]



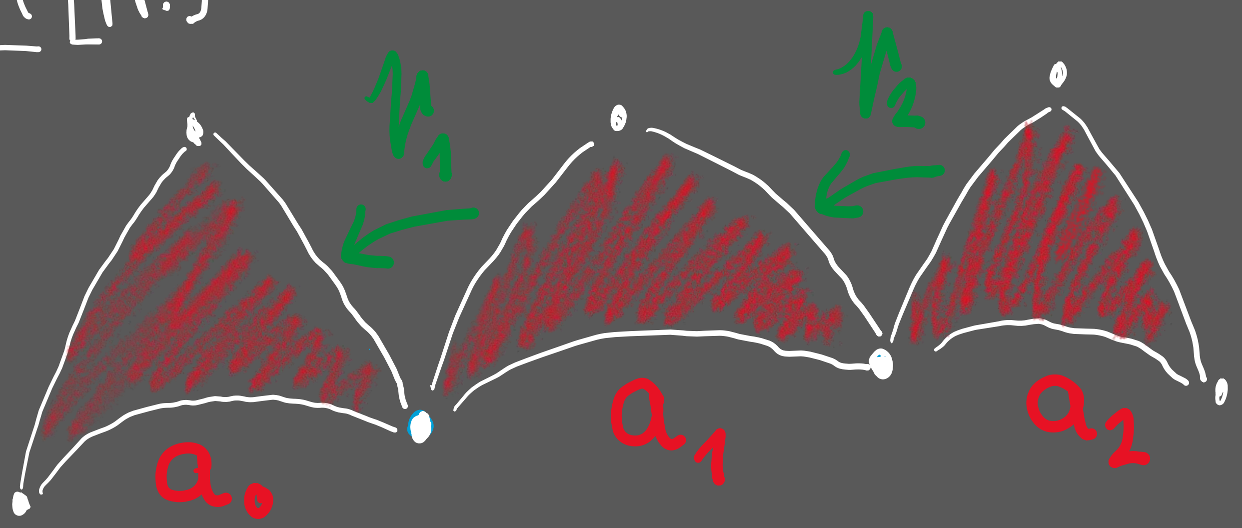
$$\sigma_i := \eta_1 + \dots + \eta_i$$

equivariant symplecto.

$$\mathrm{Rep}^{OT}_{\alpha}(\Sigma_{0,n}, PSL_2\mathbb{R}) \longrightarrow \mathbb{CP}^{n-3}$$

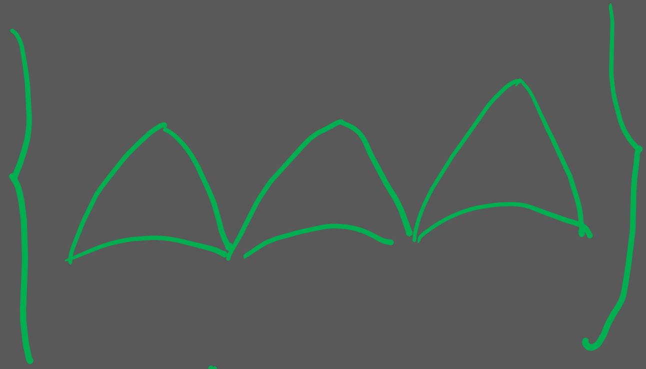
$$[\phi] \longmapsto \left[\sqrt{a_0} : e^{i\sigma_1}\sqrt{a_1} : \dots : e^{i\sigma_{n-3}}\sqrt{a_{n-3}}\right]$$

THM [M.]



$$\sigma_i := r_1 + \dots + r_i$$

$$\omega_{\text{Goldman}} = \frac{1}{2} \sum_{i=1}^{n-3} da_i \wedge d\sigma_i$$



hyperbolic metrics on $\Sigma_{0,n}$ with conical singularities

$$\mathrm{Rep}^{DT}_{\alpha}(\Sigma_{0,n}, PSL_2\mathbb{R})$$

$$\mathbb{CP}^{n-3}$$

[Deroin-Tholozan 19']

[M. 21']

further remarks:

(1) $\exists$ generalization for $G = SU(p,q), Sp(2n,\mathbb{R}), \dots$

[Thobzan-Toulisse]

(2) $\mathrm{Mod}(\Sigma_{0,n}) \curvearrowright \mathrm{Rep}^{DT}_{\alpha}(\Sigma_{0,n}, PSL_2\mathbb{R})$

[M.]

ergodic