

ON REMARKABLE REPRESENTATIONS

$$\pi_1 \left(\text{[Diagram of a genus } g \text{ surface]} \right) \rightarrow \text{PSL}_2 \mathbb{R}$$

Arnaud Malet
University of Heidelberg

Geometry & Topology seminar
University of Bristol

Our plan today

- (1) Character varieties
- (2) Dehn-Thurston representations
- (3) A combinatorial model

(1) Character varieties

closed
surface

lie group

(eg: $G = \mathrm{PSL}_2(\mathbb{R})$)

$$\mathrm{Hom}(\pi_1(\Sigma_g), G)$$

closed
surface

lie group

(eg: $G = \text{PSL}_2(\mathbb{R})$)

$$\text{Hom}(\pi_1(\Sigma_g), G)$$

algebraic variety
inside G^{2g}

closed
surface

lie group

(eg: $G = \mathrm{PSL}_2(\mathbb{R})$)

$$\mathrm{Hom}(\pi_1(\Sigma_g), G) \hookrightarrow \mathrm{Inn}(G)$$

algebraic variety
inside G^{2g}

$$\text{Rep}(\Sigma_g, G) := \text{Hom}(\pi_1(\Sigma_g), G) / \text{Inn}(G)$$

↖ character variety
of (Σ_g, G)

punctured
surface

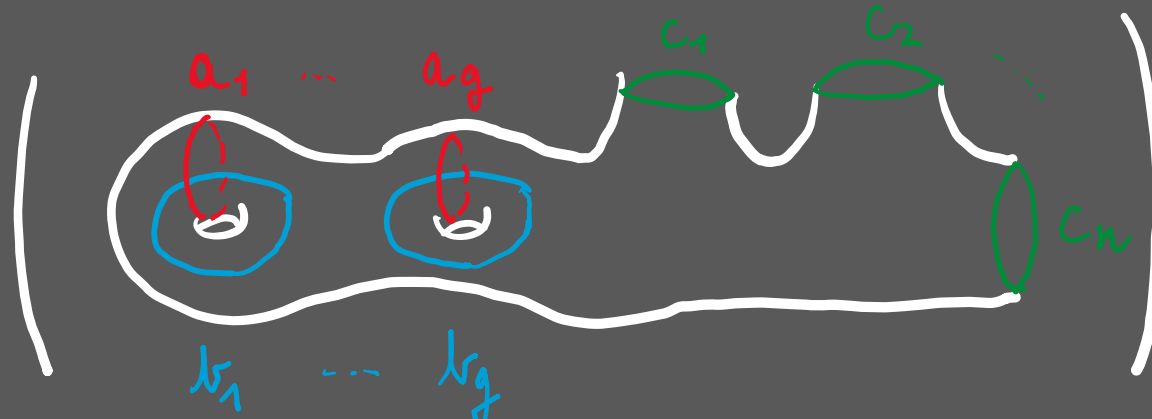
lie group

$$\text{Hom}(\pi_1(\Sigma_{g,n}), \mathfrak{g})$$

punctured
surface

lie group

$$\text{Hom}(\pi_1(\Sigma_{g,n}), G)$$



The diagram shows a genus g surface with n punctures. The surface is represented as a horizontal strip with g handles. Each handle i has a red loop a_i and a blue loop b_i . The punctures are represented by green loops c_j at the right end of the surface. The labels a_1, \dots, a_g are in red, b_1, \dots, b_g are in blue, and c_1, c_2, \dots, c_n are in green. The entire diagram is enclosed in large parentheses.

$$\pi_1 \left(\text{Surface} \right) = \left\langle a_i, b_i, c_j \mid \prod_{i=1}^g [a_i, b_i] = \prod_{j=1}^n c_j \right\rangle$$

punctured
Surface

lie group

$$\text{Hom}(\pi_1(\Sigma_{g,n}), \mathfrak{g})$$

The diagram shows a white outline of a surface with g handles and n punctures. Red loops labeled a_1, \dots, a_g encircle the handles. Blue loops labeled b_1, \dots, b_g encircle the handles from the bottom. Green loops labeled c_1, c_2, \dots, c_n encircle the punctures. The entire diagram is enclosed in large parentheses, followed by an isomorphism symbol \cong and the free group F_{2g+n-1} .

$$\pi_1 \left(\text{Surface with } g \text{ handles and } n \text{ punctures} \right) \cong F_{2g+n-1}$$

punctured
surface

lie group

$$\text{Hom}(\pi_1(\Sigma_{g,n}), \mathfrak{g}) \cong \mathfrak{g}^{2g+n-1}$$

The diagram shows a genus g surface with n punctures. The fundamental group π_1 is represented by a set of generators: a_1, \dots, a_g (red circles), b_1, \dots, b_g (blue circles), and c_1, \dots, c_n (green circles). The surface is shown as a horizontal strip with g handles and n punctures. The generators are labeled as follows: a_i and b_i are loops around the i -th handle, and c_i are loops around the i -th puncture.

$$\pi_1 \left(\text{Surface} \right) \cong F_{2g+n-1}$$

punctured
surface

lie group

$$\text{Hom}_g(\pi_1(\Sigma_{g,n}), G)$$

collection of n
conjugacy classes $C_1, \dots, C_n \in G/\text{conj}$

$\phi(c_i) \in C_i$

punctured
surface

lie group

$$\text{Hom}_g(\pi_1(\Sigma_{g,n}), G) \cong \text{Inn}(G)$$

collection of n
conjugacy classes $C_1, \dots, C_n \in G/\text{conj}$

$\phi(C_i) \in C_i$

$$\text{Rep } \rho(\Sigma_{g,n}, G) := \text{Hom } \rho(\pi_1(\Sigma_{g,n}), G) / \text{Inn}(G)$$



relative character
variety of $(\Sigma_{g,n}, G)$

main task: study

↙ the topology / geometry
of (relative) character varieties

$\text{Rep}_e(\Sigma_{g,n}, G)$

main task: study

↙ the topology / geometry
of (relative) character varieties

$\text{Rep}_e(\Sigma_{g,n}, G)$

(1) Goldman symplectic
structure

main task: study

↙ the topology / geometry
of (relative) character varieties

$\text{Rep}_e(\Sigma_{g,n}, G)$

(1) Goldman symplectic
structure

(2) Toledo number

even dimensional
manifold

symplectic
manifold

(M, ω)

closed & non-degenerate
2-form

even dimensional
manifold

symplectic
manifold

(M, ω)

closed & non-degenerate
2-form

section of
 T^*M

ω
 \longleftrightarrow

section of
 TM

even dimensional
manifold

symplectic
manifold

(M, ω)

closed & non-degenerate
2-form

smooth
function



section of
 T^*M



section of
 TM

$$f: M \rightarrow \mathbb{R}$$

df



even dimensional
manifold

symplectic
manifold

(M, ω)

closed & non-degenerate
2-form

smooth
function



section of
 T^*M

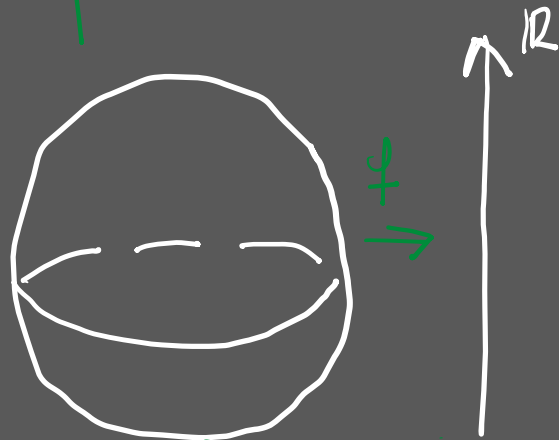


section of
 TM



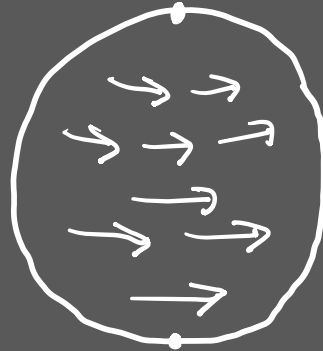
flow

$f: M \rightarrow \mathbb{R}$

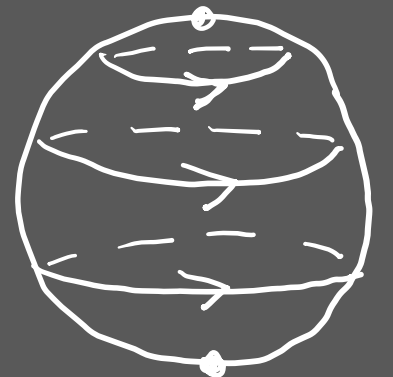


df

X_f



$\Phi_f^t: M \rightarrow M, t \in \mathbb{R}$



$$\text{Tol} : \text{Rep}(\Sigma_{g,n}, G) \longrightarrow \mathbb{R}$$

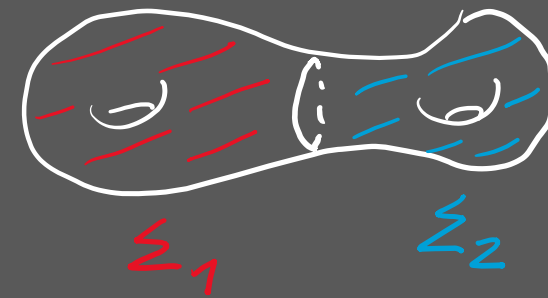
Tolendo number
[Burger-Iozzi-Wienhard]

* continuous

* locally constant on relative character variety

* (additivity)

$$\text{Tol}(\phi) = \text{Tol}(\phi|_{\pi_1(\Sigma_1)}) + \text{Tol}(\phi|_{\pi_1(\Sigma_2)})$$



* (Milnor-Wood inequality)

$$|\text{Tol}| \leq |\chi(\Sigma_{g,n})| \cdot \text{rank}(G)$$

* (...)

(relative) character
variety

= Symplectic manifold
associated to
 $(\Sigma_{g,n}, G)$

$(\text{Rep}_e(\Sigma_{g,n}, G), \omega_{\text{Goldman}})$

(2) Denzin - Thoburn representations

A remarkable example: Dehn-Thurston representations

surface = $\Sigma_{0,n}$

lie group = $PSL_2\mathbb{R}$



$Rep_{\alpha}^{DT}(\Sigma_{0,n}, PSL_2\mathbb{R})$

\subseteq

$Rep_{\alpha}(\Sigma_{0,n}, PSL_2\mathbb{R})$

compact component

elliptic conjugacy classes

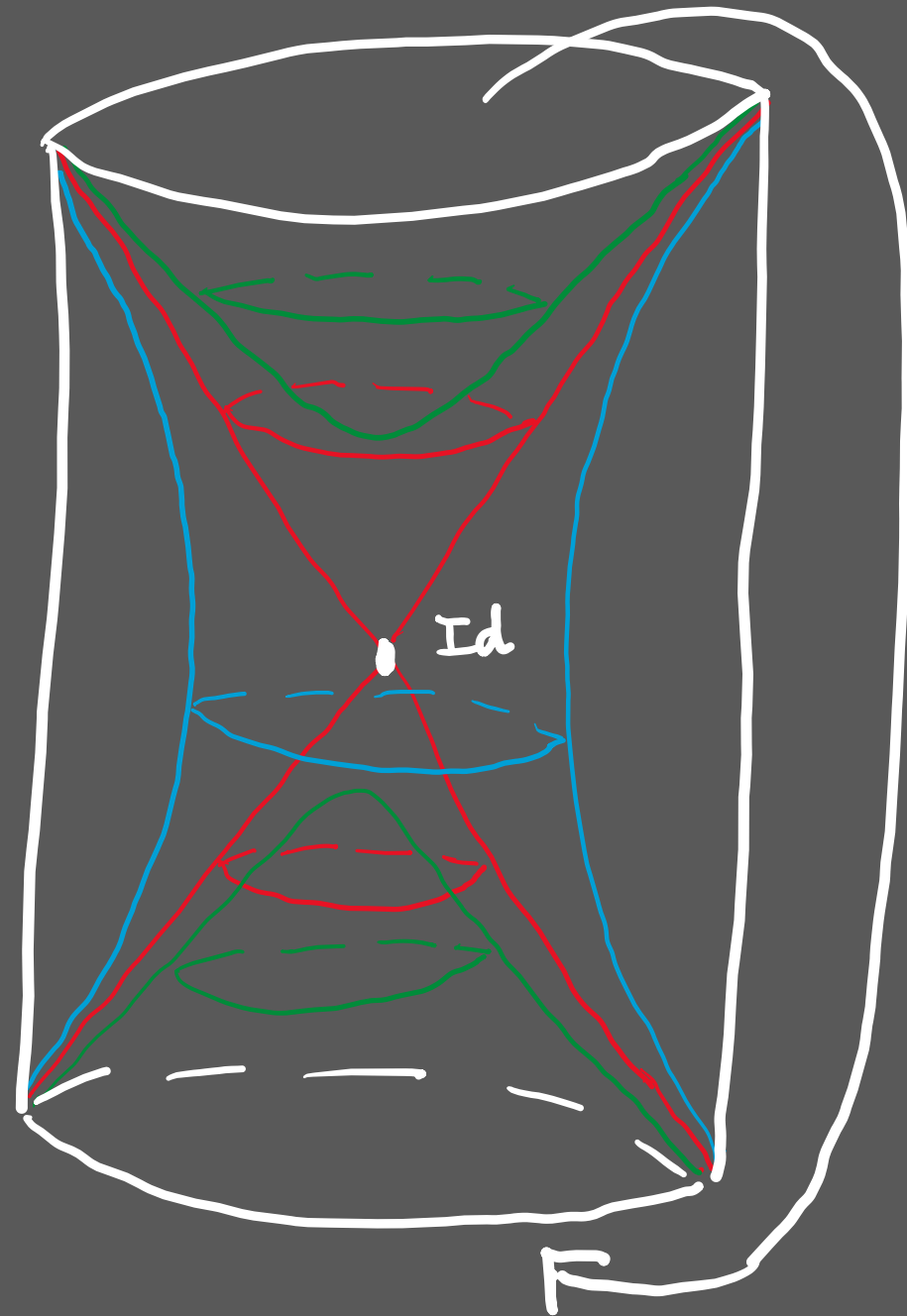
$PSL_2 \mathbb{R}$

elliptic = { unique fixed
point in \mathbb{H}^2 }

$\approx \mathbb{H}^2 \times (0, 2\pi)$

fixed
point

ν = angle
of rotation



$$\alpha = (\alpha_1, \dots, \alpha_n) \in (0, 2\pi)^n$$

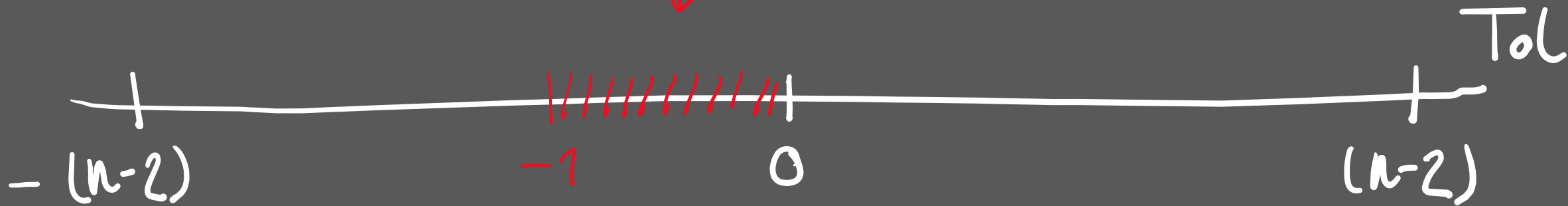
crucial
angles

$$2\pi \cdot n > \alpha_1 + \dots + \alpha_n > 2\pi \cdot (n-1)$$

condition

$$\text{Rep}_\alpha^{\text{OT}}(\Sigma_{0,n}, \text{PSL}_2\mathbb{R}) \subseteq \text{Rep}_\alpha(\Sigma_{0,n}, \text{PSL}_2\mathbb{R})$$

$T_d \in (-1, 0)$



THM [Dehn - Tholozan]

$$\left(\text{Rep}_{\mathbb{Z}}^{\text{OT}}(\Sigma_{0,n}, \text{PSL}_2\mathbb{R}), W_{\text{Goldman}} \right) \cong \left(\mathbb{C}P^{n-3}, \lambda \cdot \omega_{\text{FS}} \right)$$

$$\begin{aligned} \lambda &= \alpha_1 + \dots + \alpha_n - 2\pi(n-1) \\ &= -\frac{1}{2\pi} \text{Td}(\phi) \end{aligned}$$

THM [Dehn - Tholozan]

$$\left(\text{Rep}_2^{\text{OT}}(\Sigma_{0,n}, \text{PSL}_2\mathbb{R}), W_{\text{Goldman}} \right) \cong \left(\mathbb{C}P^{n-3}, \lambda \cdot \omega_{\text{FS}} \right)$$

↑
compact

$$\begin{aligned} \lambda &= \alpha_1 + \dots + \alpha_n - 2\pi(n-1) \\ &= -\frac{1}{2\pi} \text{Td}(\phi) \end{aligned}$$

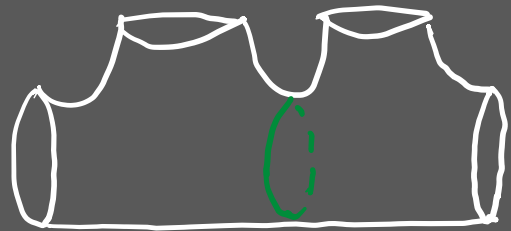
THM [Dehn - Tholozan]

$$\left(\text{Rep}_2^{\text{OT}}(\Sigma_{0,n}, \text{PSL}_2\mathbb{R}), W_{\text{Goldman}} \right) \cong \left(\mathbb{CP}^{n-3}, \lambda \cdot W_{\text{FS}} \right)$$

compact

totally elliptic

$$\begin{aligned} \lambda &= \alpha_1 + \dots + \alpha_n - 2\pi(n-1) \\ &= -\frac{1}{2\pi} \text{Td}(\phi) \end{aligned}$$

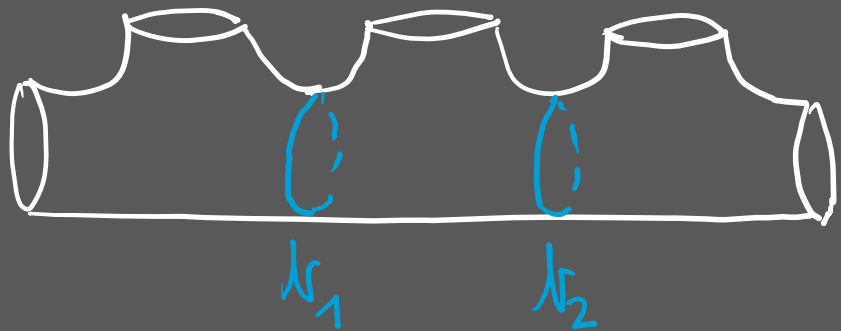


elliptic

proof: $T^{n-3} \hookrightarrow (\text{Rep}_\alpha^{\text{OT}}, W_{\text{Goldman}})$

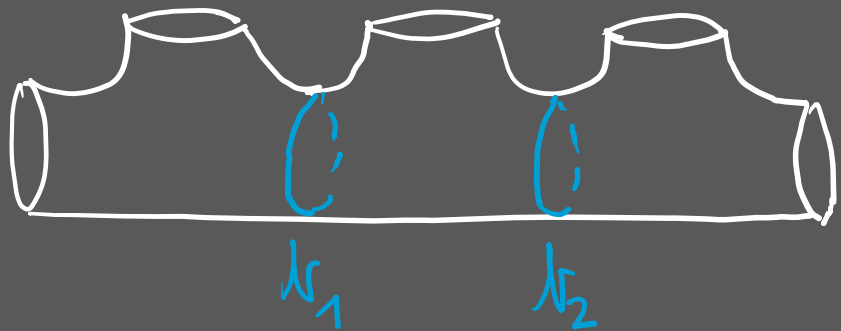
proof: $T^{n-3} \hookrightarrow (\text{Rep}_\alpha^{\text{OT}}, W_{\text{Goldman}})$

($n=5$)



proof: $T^{n-3} \hookrightarrow (\text{Rep}_{\alpha}^{\text{OT}}, \mathcal{W}_{\text{Goldman}})$

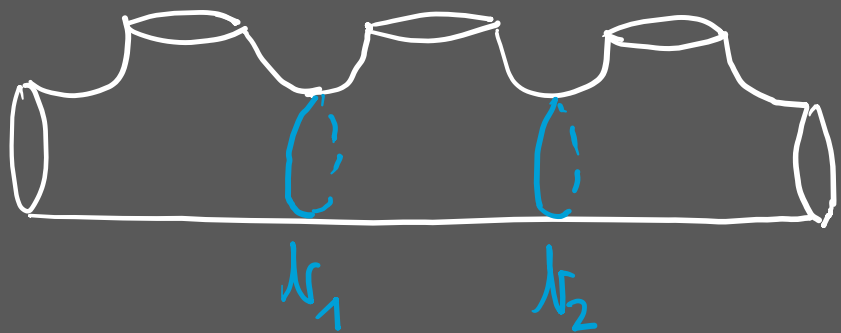
($n=5$)



$\rightsquigarrow \forall \gamma_i: \text{Rep}_{\alpha}^{\text{OT}} \rightarrow (0, 2\pi)$

proof: $T^{n-3} \hookrightarrow (\text{Rep}_\alpha^{\text{OT}}, W_{\text{Goldman}})$

($n=5$)



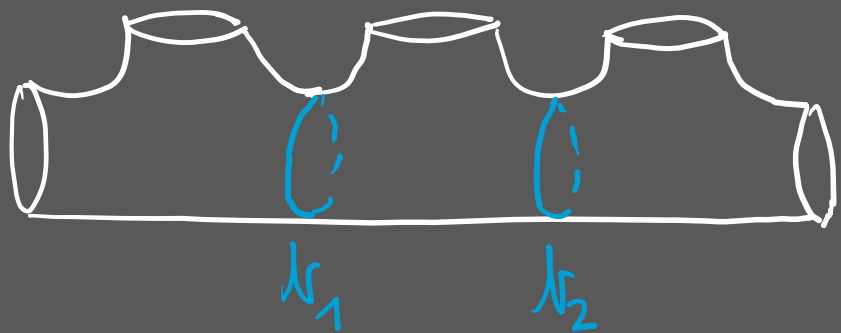
$\rightsquigarrow \forall \alpha_i: \text{Rep}_\alpha^{\text{OT}} \rightarrow (0, 2\pi)$

\rightsquigarrow periodic & commutative

flows $\Phi_{\alpha_i}: \mathbb{R} \times \text{Rep}_\alpha^{\text{OT}} \rightarrow \text{Rep}_\alpha^{\text{OT}}$

proof: $T^{n-3} \hookrightarrow (\text{Rep}_\alpha^{\text{OT}}, W_{\text{Goldman}})$

($n=5$)



$\rightsquigarrow \forall \alpha_i: \text{Rep}_\alpha^{\text{OT}} \rightarrow (0, 2\pi)$

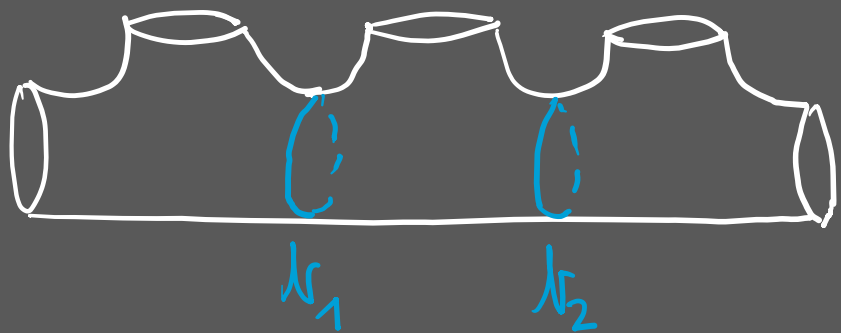
\rightsquigarrow periodic & commutative

flows $\Phi_{\alpha_i}: \mathbb{R} \times \text{Rep}_\alpha^{\text{OT}} \rightarrow \text{Rep}_\alpha^{\text{OT}}$

$= T^{n-3} \hookrightarrow (\text{Rep}_\alpha^{\text{OT}}, W_{\text{Goldman}})$

proof: $T^{n-3} \hookrightarrow (\text{Rep}_\alpha^{\text{OT}}, W_{\text{Goldman}})$

($n=5$)



$\rightsquigarrow \forall r_i: \text{Rep}_\alpha^{\text{OT}} \rightarrow (0, 2\pi)$

\rightsquigarrow periodic & commutative

flows $\Phi_{r_i}: \mathbb{R} \times \text{Rep}_\alpha^{\text{OT}} \rightarrow \text{Rep}_\alpha^{\text{OT}}$

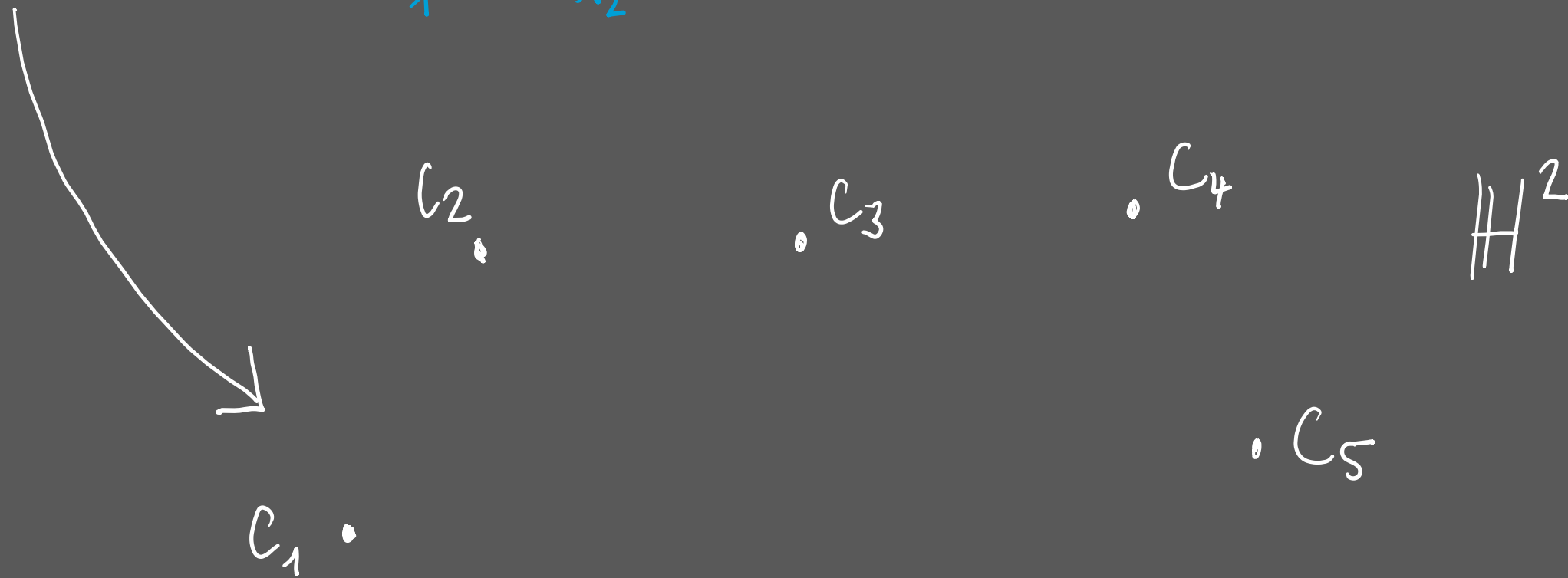
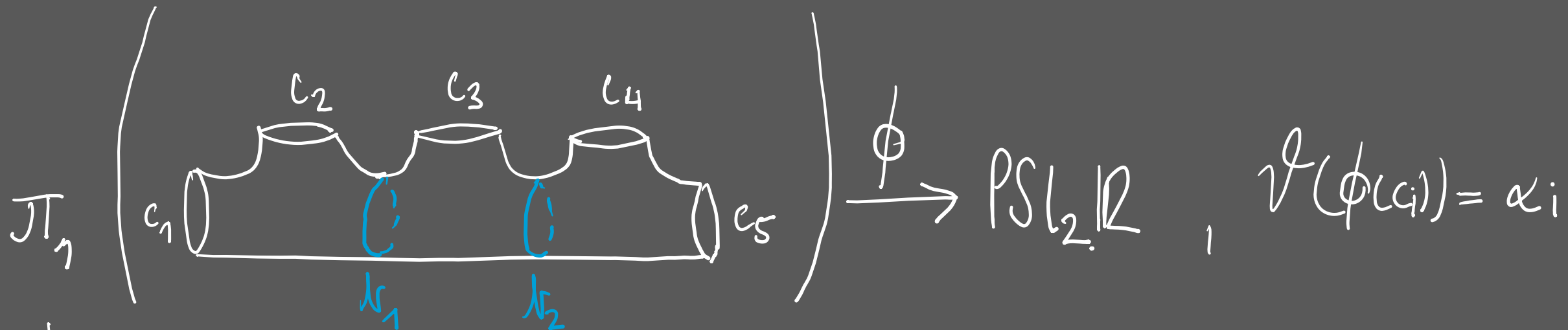
$= T^{n-3} \hookrightarrow (\text{Rep}_\alpha^{\text{OT}}, W_{\text{Goldman}})$

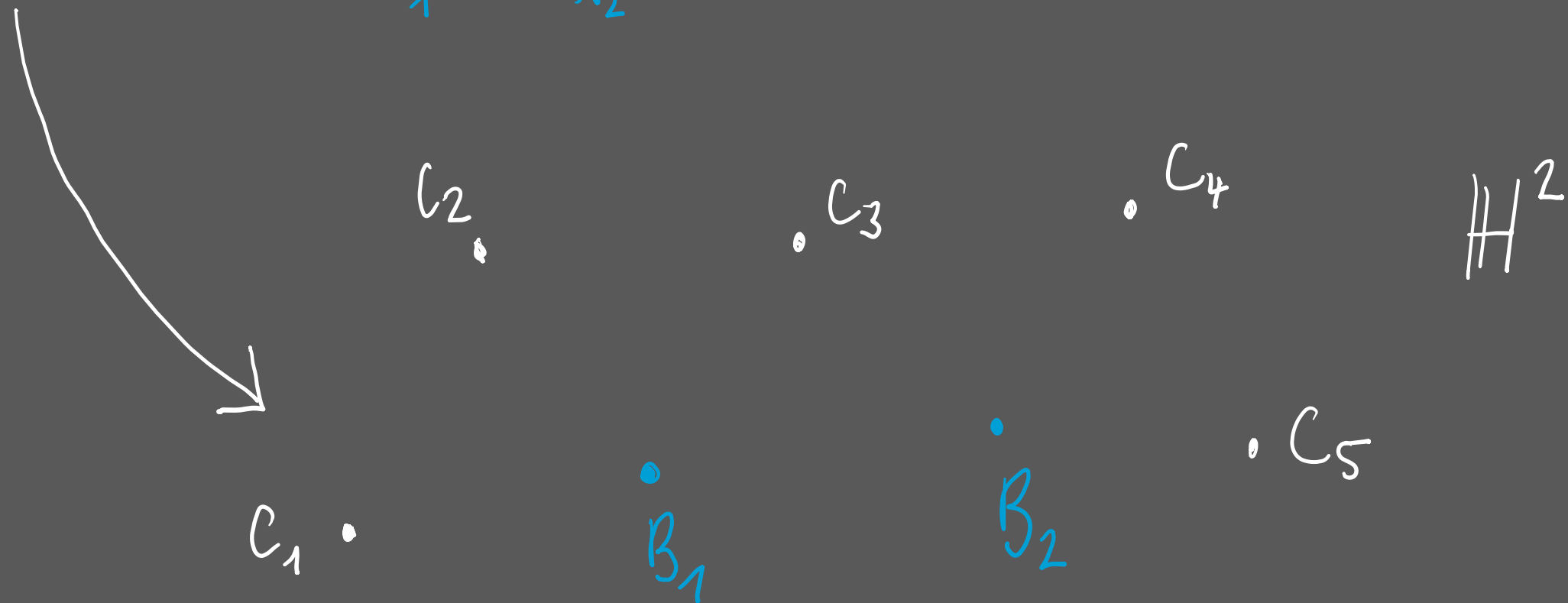
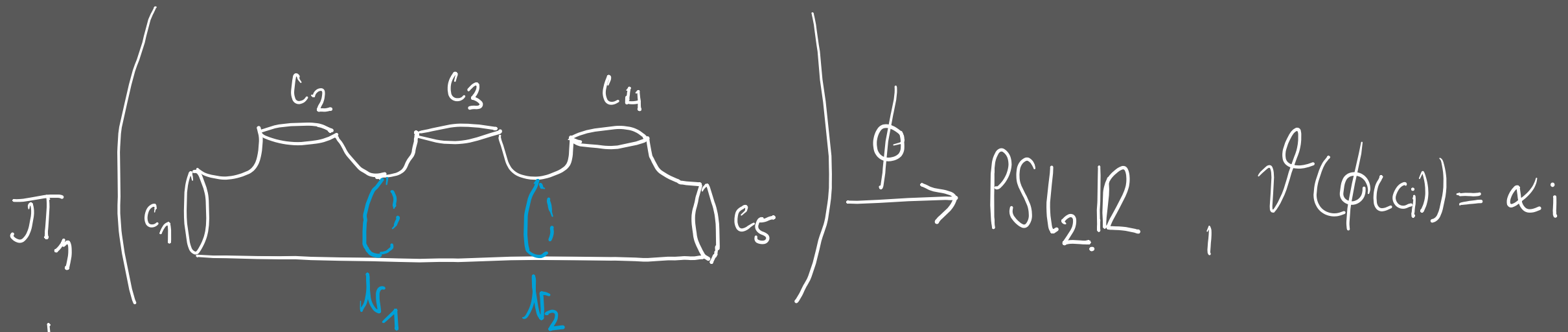
Nelzant's classification $\Rightarrow \text{Rep}_\alpha^{\text{OT}} \cong \mathbb{C}P^{n-3}$

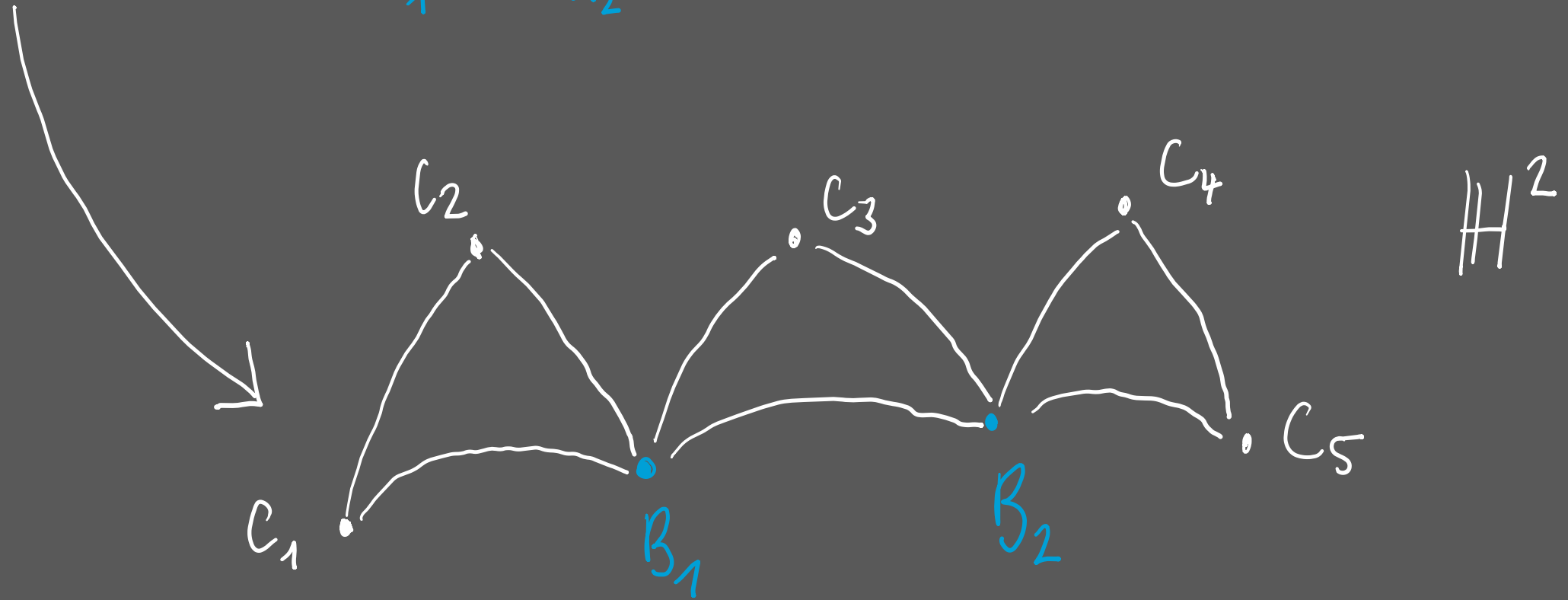
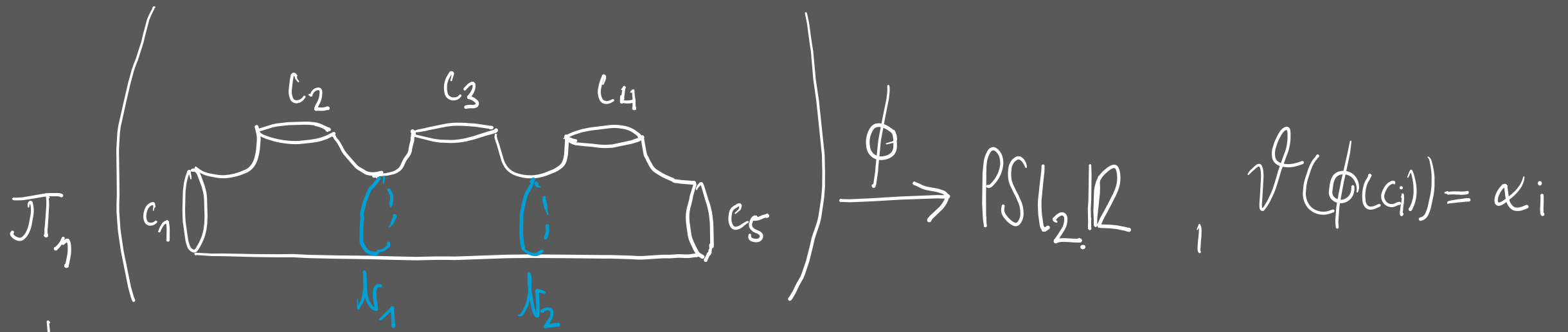
□

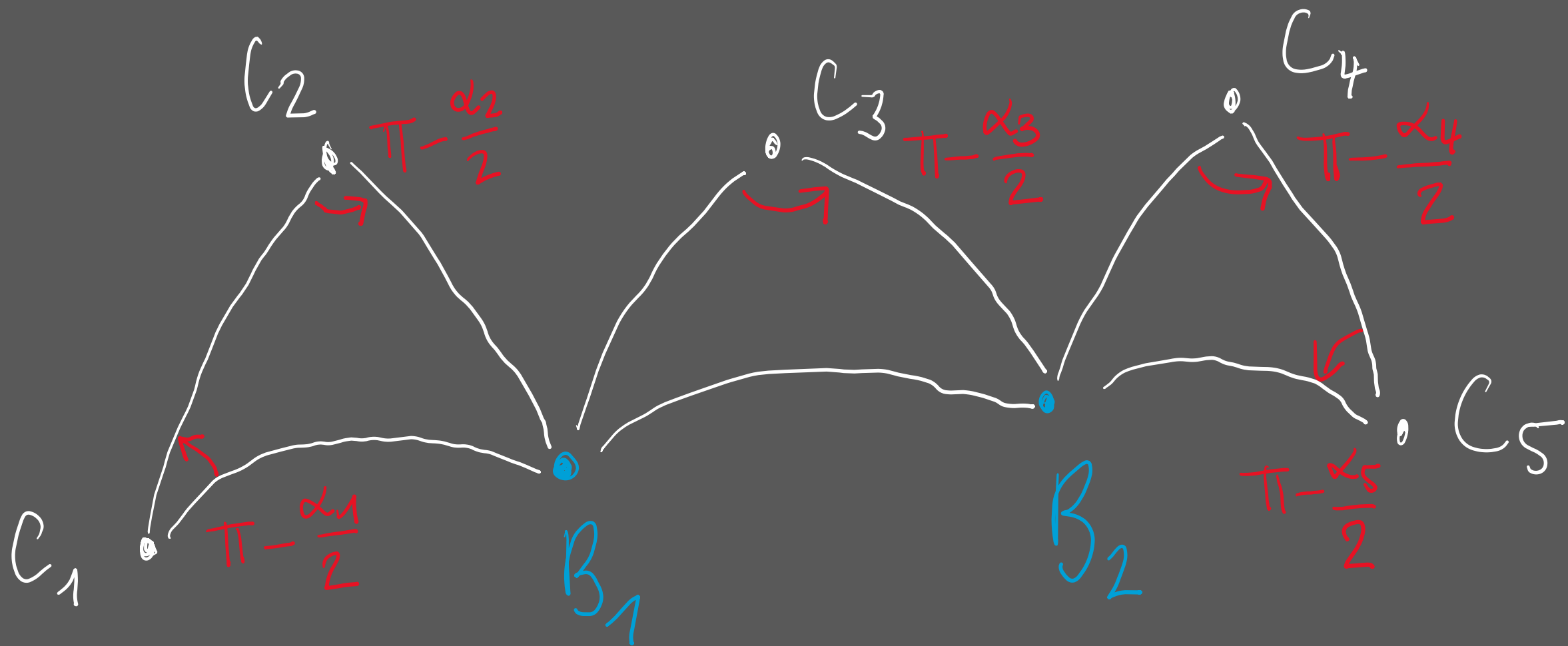
(3) A combinatorial model

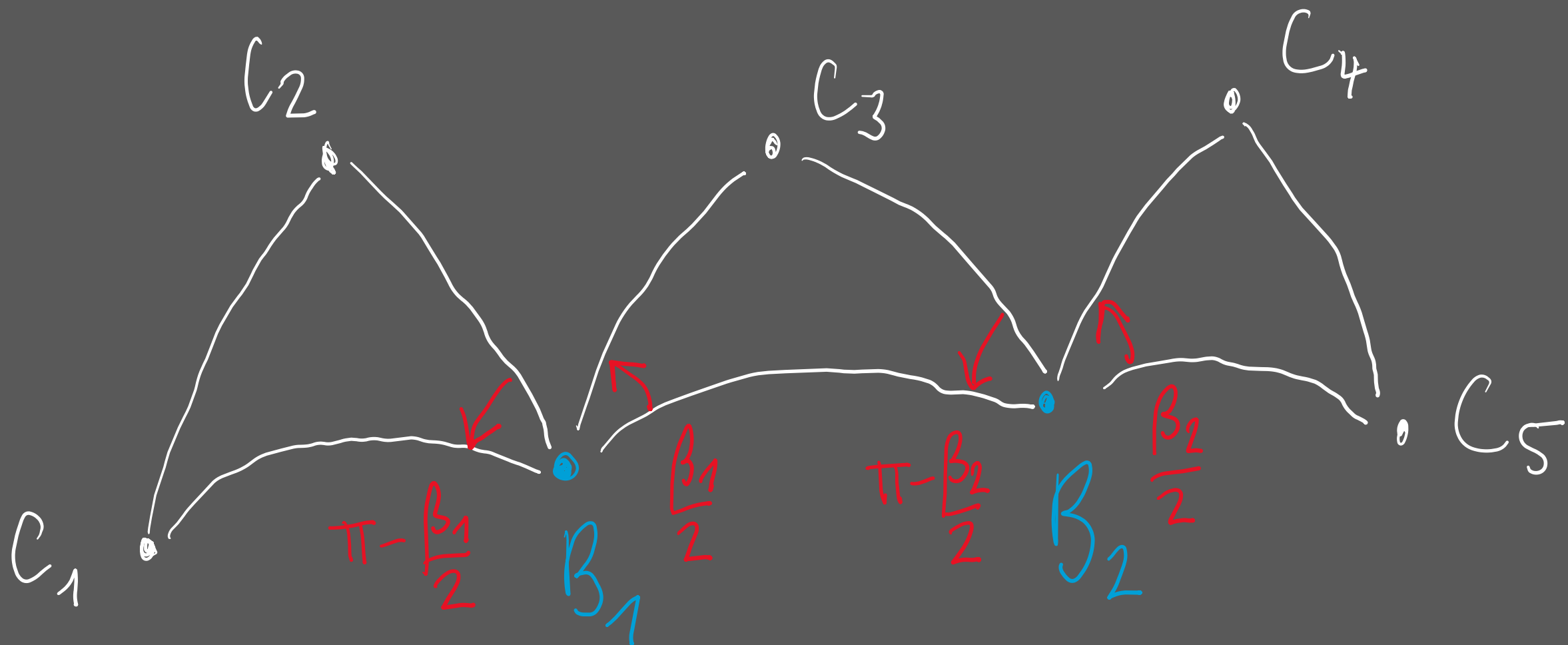
$$\mathcal{J}\mathcal{T}_\eta \left(\begin{array}{c} \text{--- } c_2 \text{ --- } c_3 \text{ --- } c_4 \text{ ---} \\ \text{--- } c_1 \text{ --- } \text{---} \text{---} \text{---} c_5 \end{array} \right) \xrightarrow{\phi} \text{PSL}_2\mathbb{R}, \quad \mathcal{V}(\phi(c_i)) = \alpha_i$$



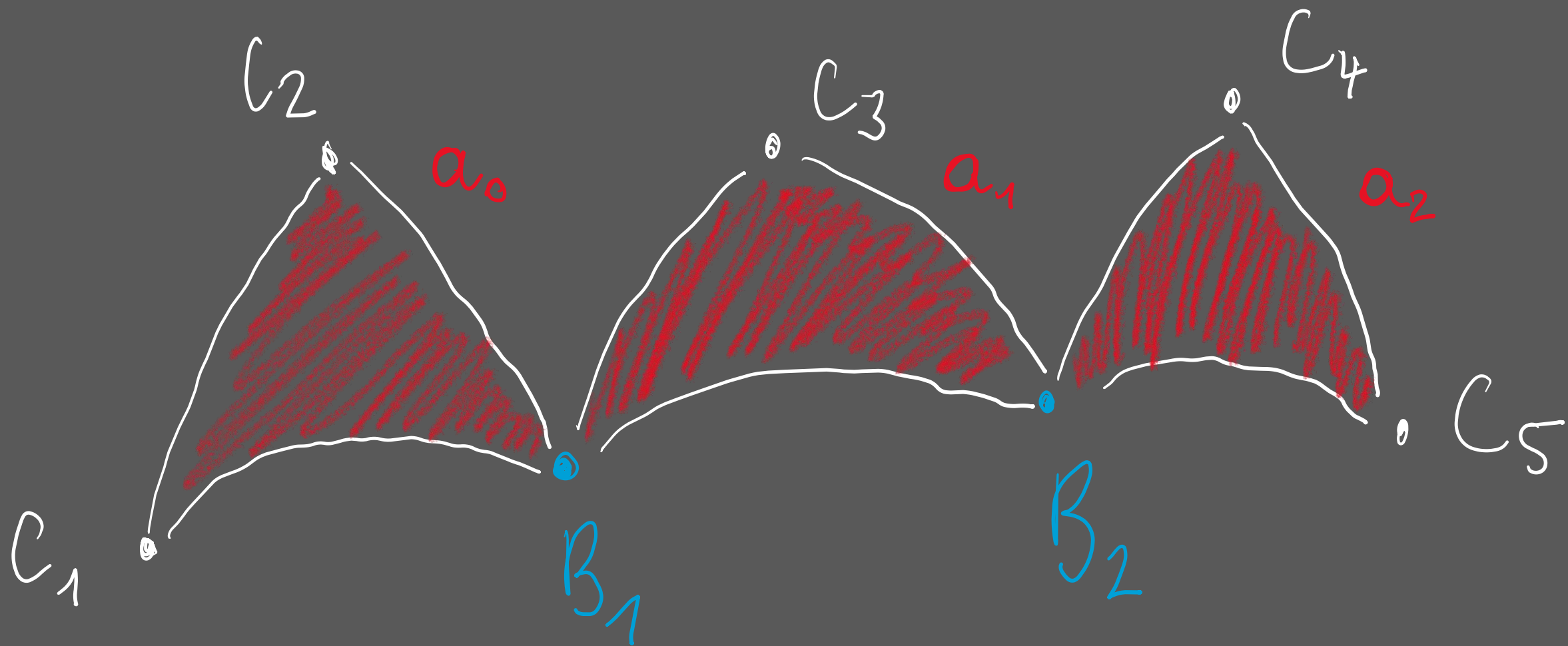




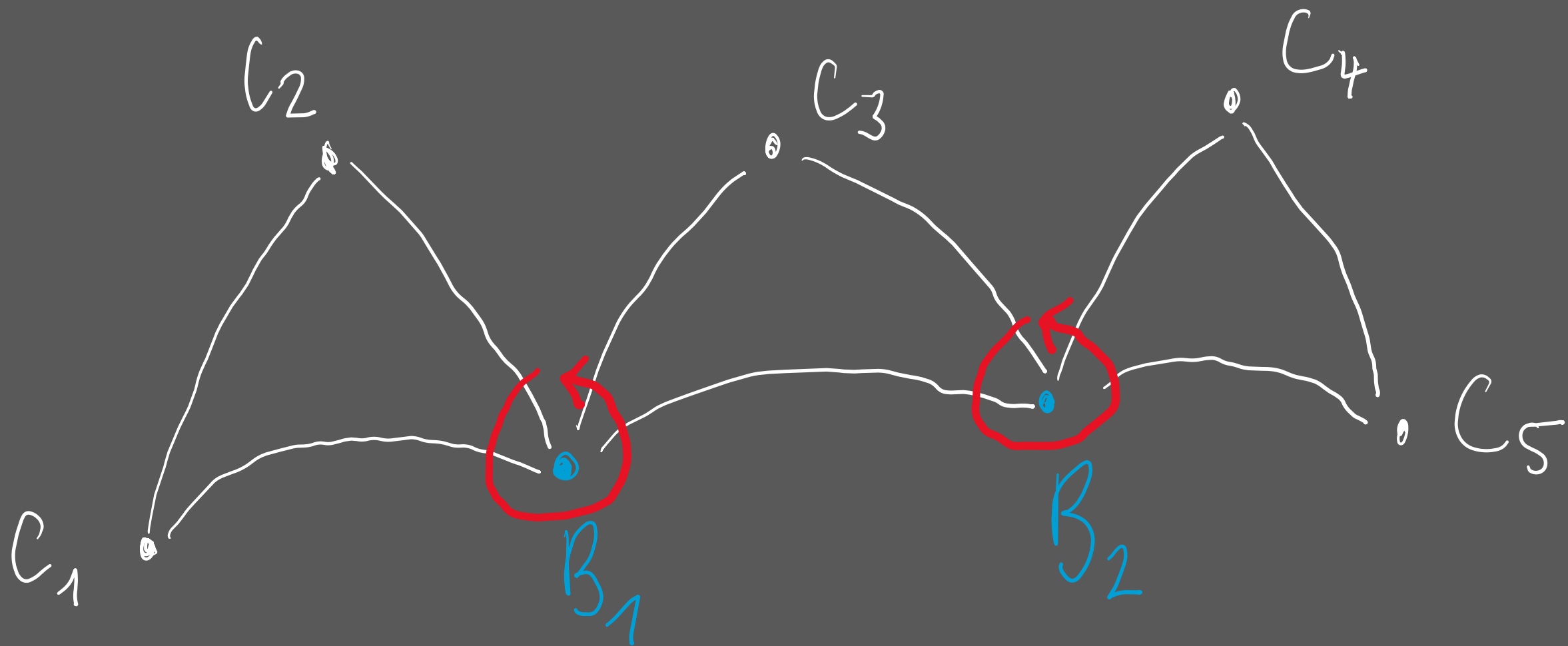


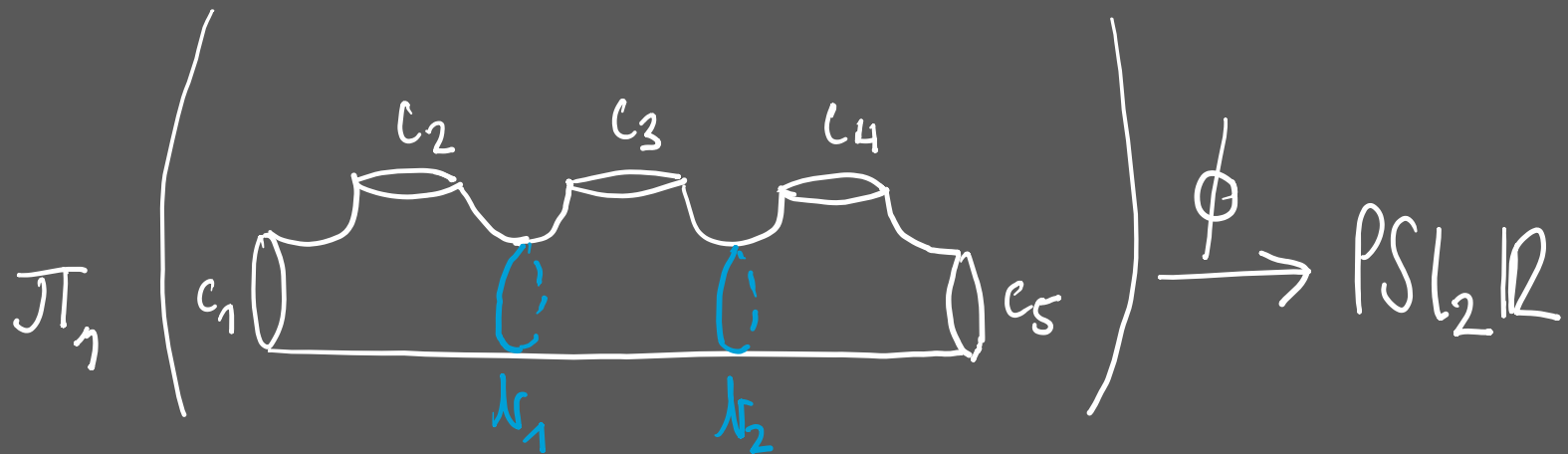


$$\beta_1 = \psi(\phi(u_1)) \quad , \quad \beta_2 = \psi(\phi(u_2))$$

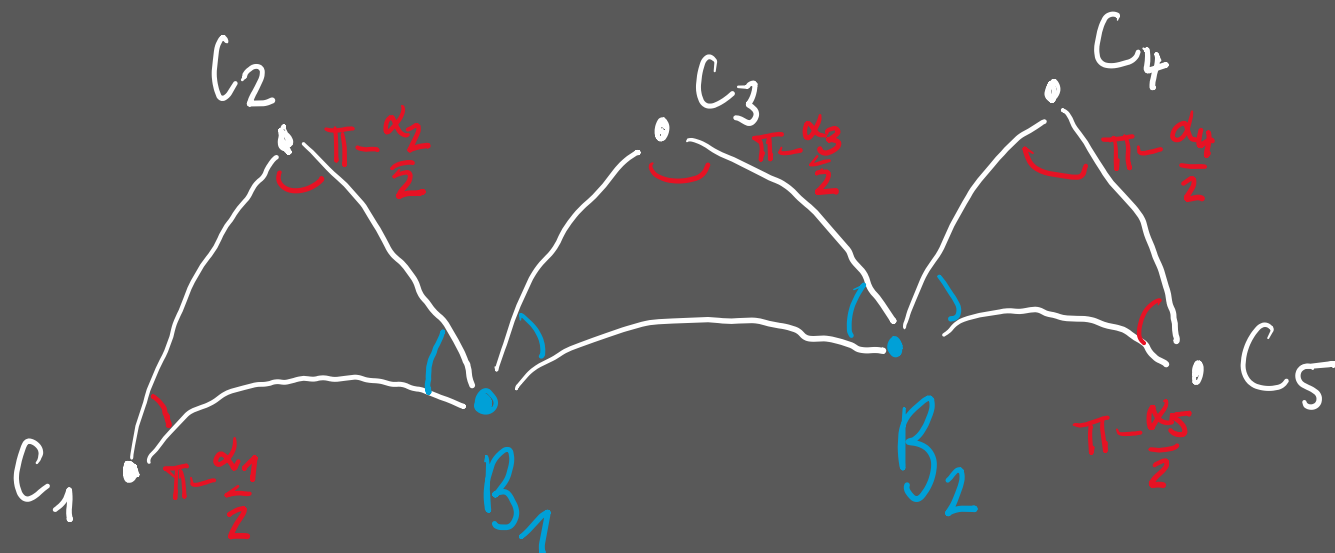


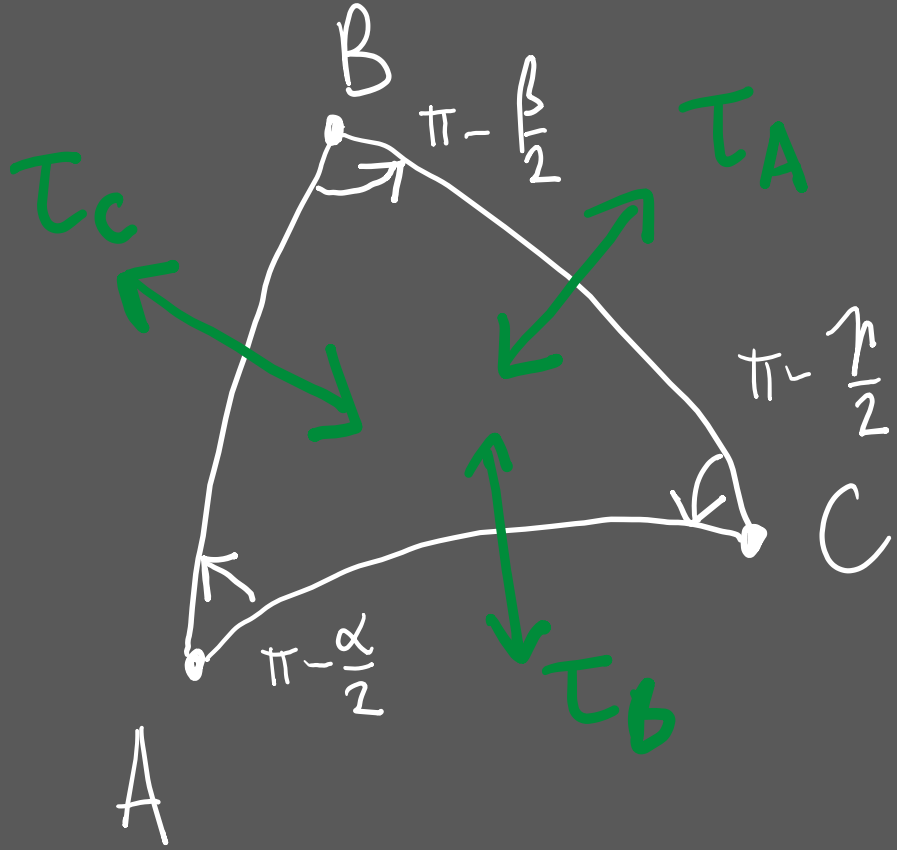
$$a_0 + a_1 + a_2 = \frac{\lambda}{2} = -\frac{1}{4\pi} \text{Td}(\phi)$$



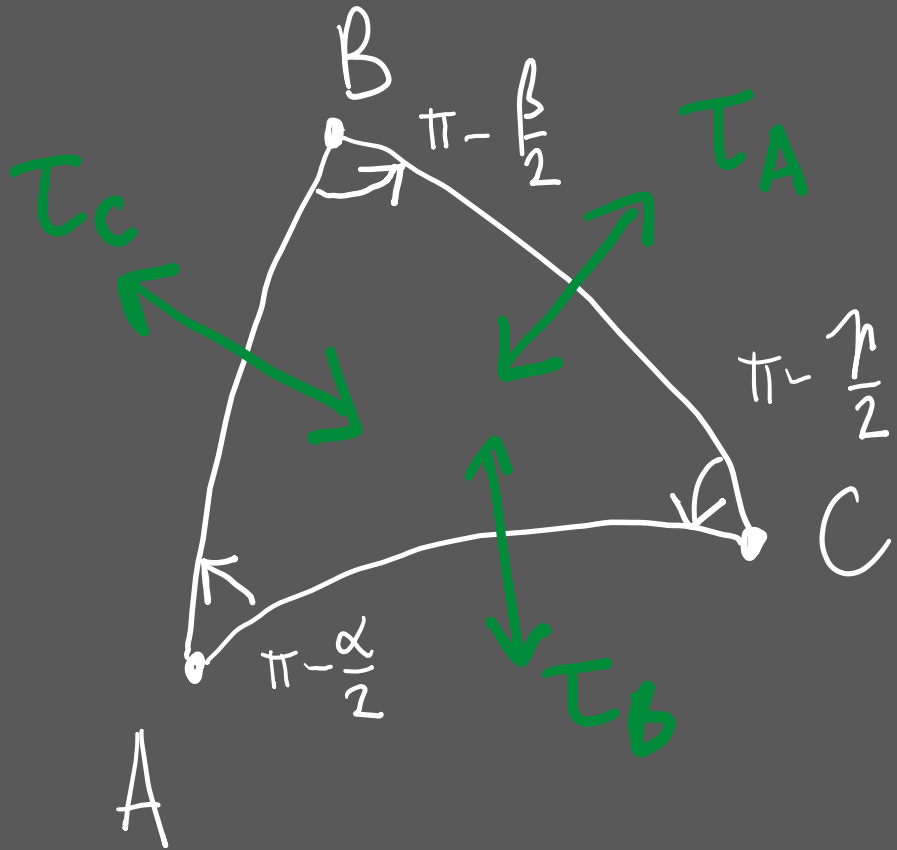


$\phi(C_i) :=$
 rotation of angle
 α_i around C_i

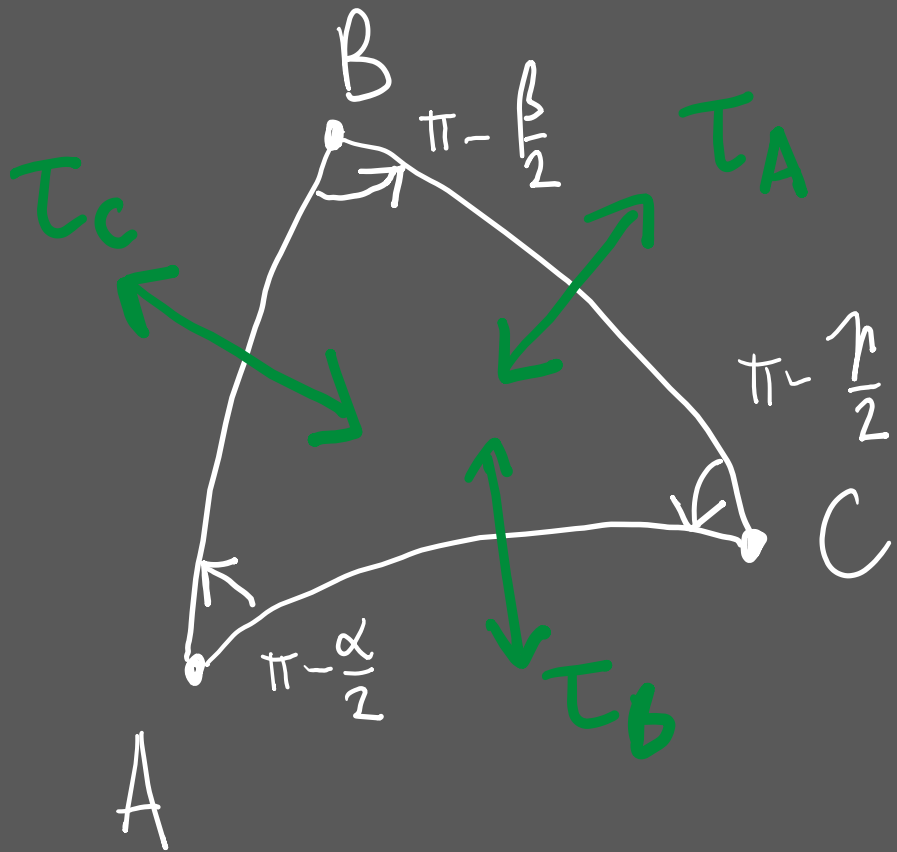




$$\tau_C \tau_A =$$



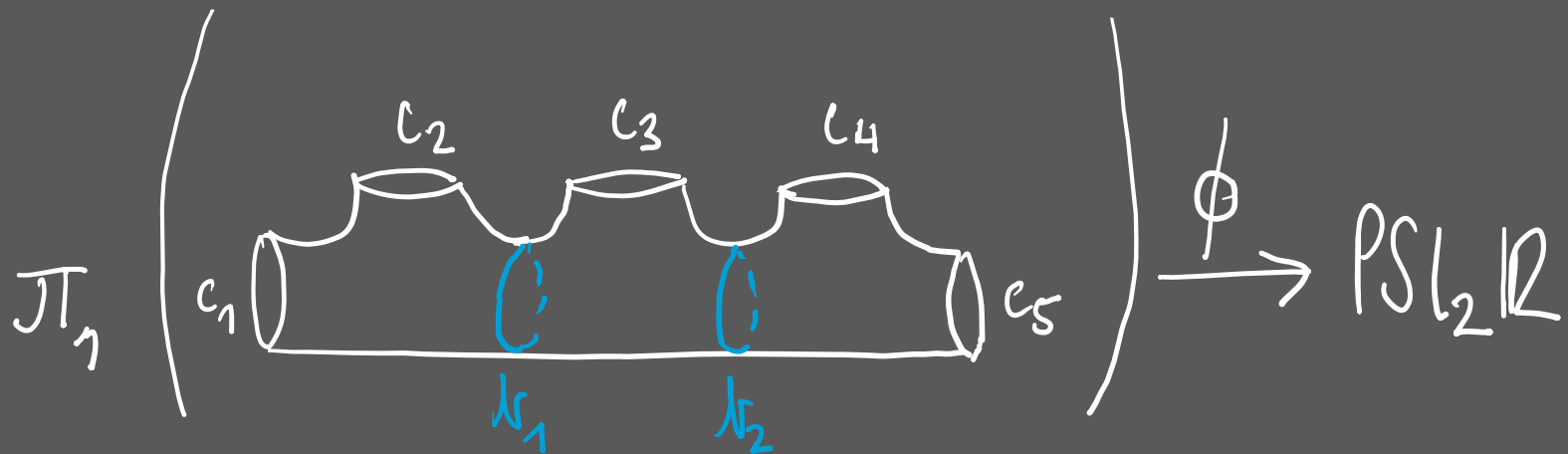
clockwise rotation
 $\tau_C \tau_A =$ of angle β
 around B



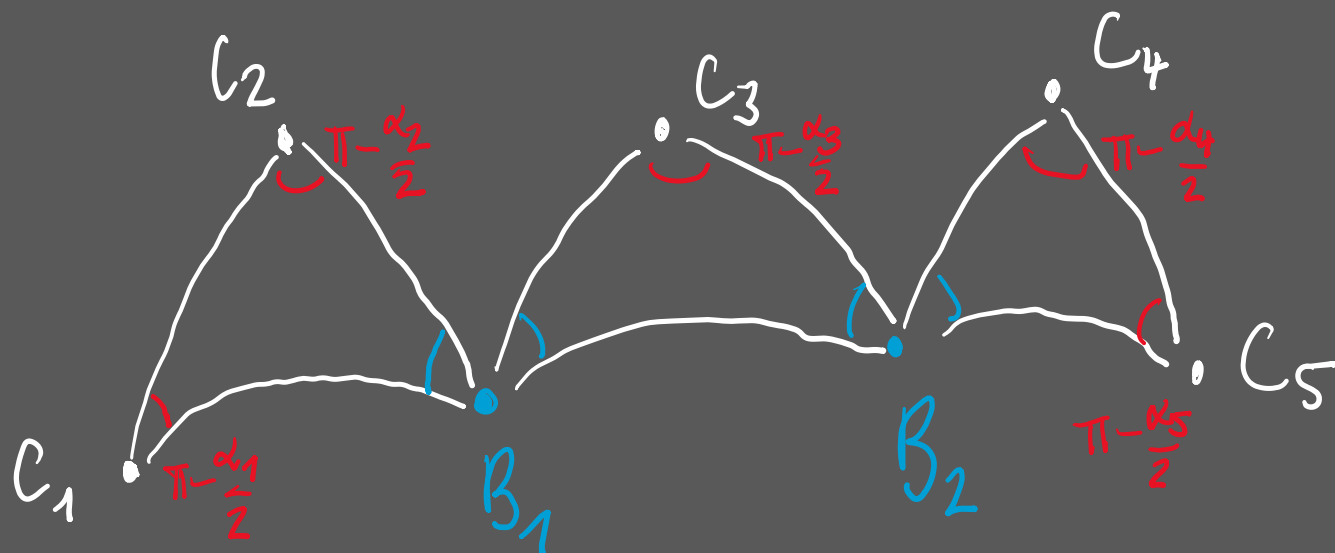
clockwise rotation

$$\tau_C \tau_A = \text{of angle } \beta \text{ around } B$$

$$(\tau_C \tau_A) \cdot (\tau_A \tau_B) \cdot (\tau_B \tau_C) = 1$$

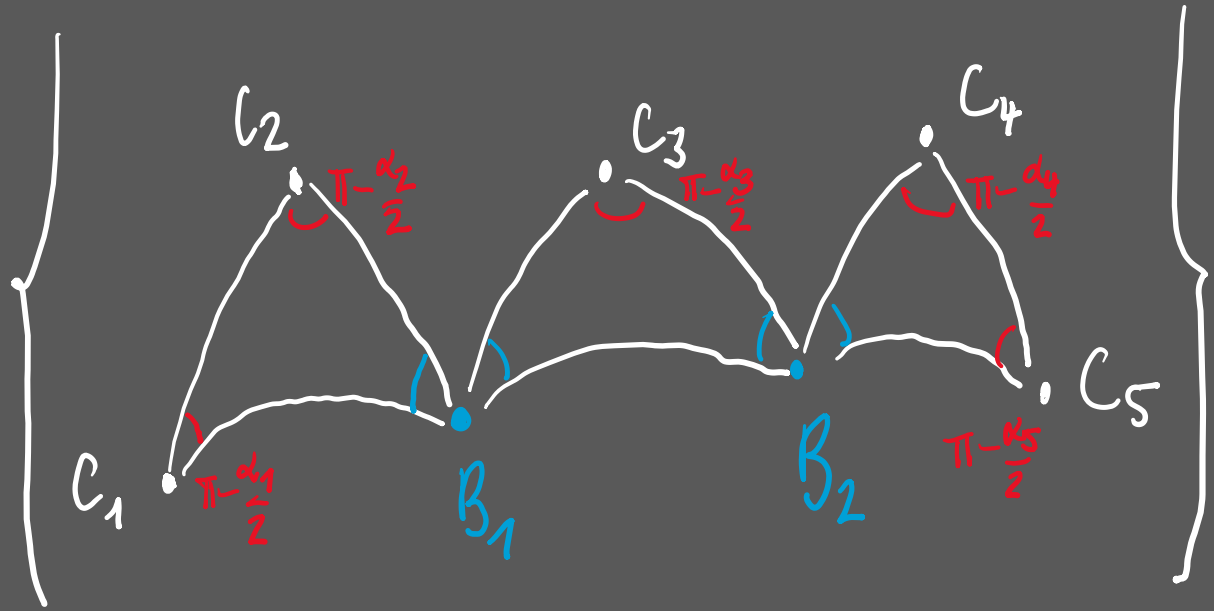


$\phi(c_i) :=$
 rotation of angle
 α_i around C_i



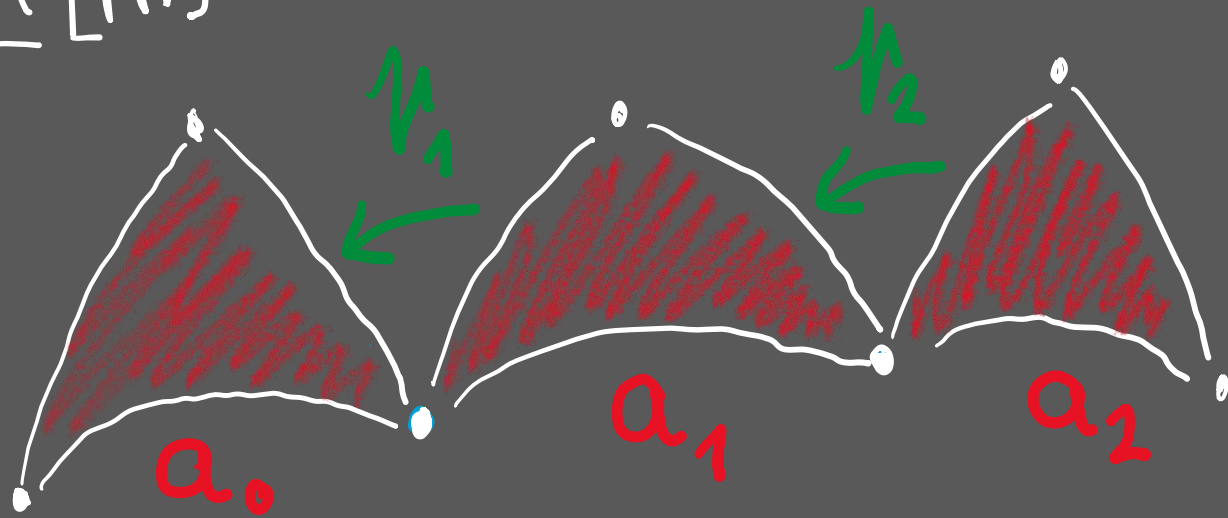
THM [M.]

$\text{Rep}^{\text{DT}}(\Sigma_{g,n}, \text{PSL}_2\mathbb{R})$



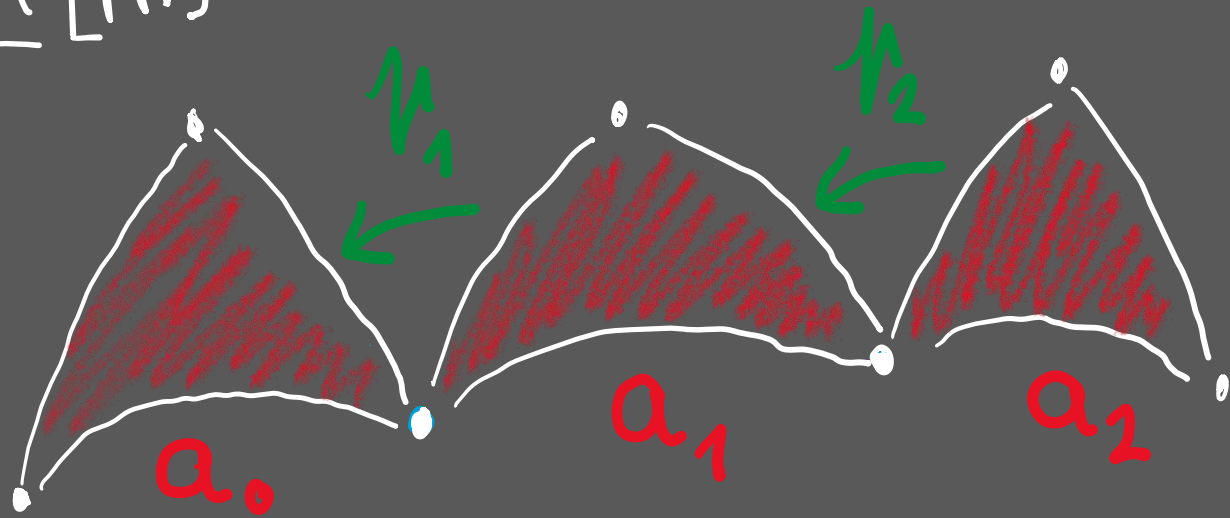
$$\frac{\mathbb{H}^n \times \mathbb{H}^{n-3}}{\text{PSL}_2\mathbb{R}}$$

THM [M.]



$$a_i := k_1 + \dots + k_i$$

THM [M.]



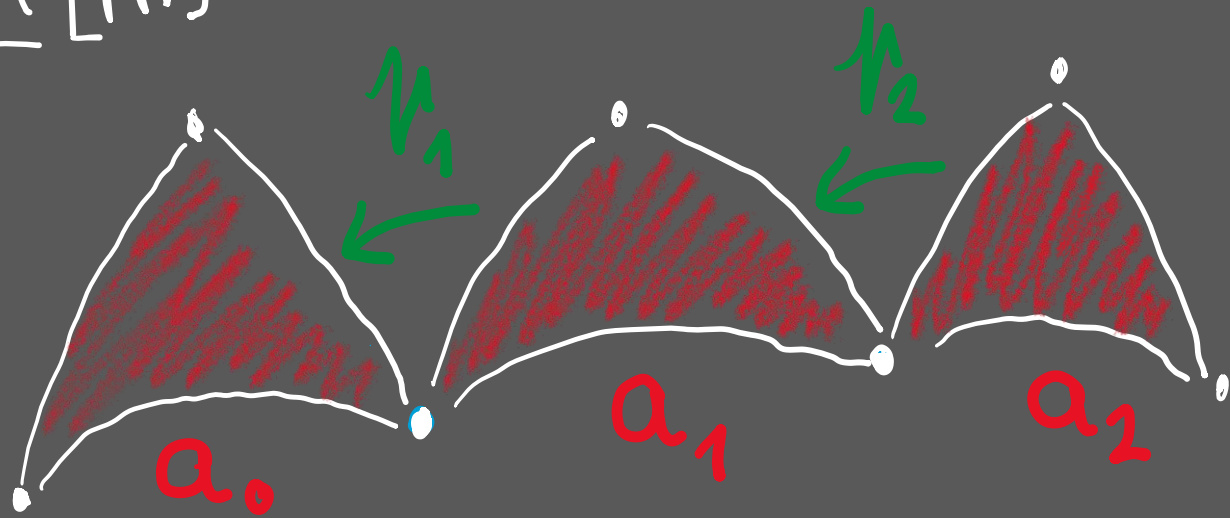
$$\sigma_i := \nu_1 + \dots + \nu_i$$

$$\text{Rep}_{\alpha}^{\text{OT}}(\Sigma_{\sigma_1, \nu_1}, \text{PSL}_2/\mathbb{R}) \longrightarrow \mathbb{C}P^{n-3}$$

equivariant
symplecto.

$$[\phi] \longmapsto [\sqrt{a_0} : e^{i\sigma_1} \sqrt{a_1} : \dots : e^{i\sigma_{n-3}} \sqrt{a_{n-3}}]$$

THM [M.]



$$\sigma_i := \nu_1 + \dots + \nu_i$$

$$W_{\text{Goldman}} = \frac{1}{2} \sum_{i=1}^{n-3} da_i \wedge d\sigma_i$$

further remarks:

(1) \exists generalization for $G = \mathrm{SU}(p, q), \mathrm{Sp}(2n, \mathbb{R}), \dots$

[Theban-Toulisse]

(2) $\mathrm{Mod}(\Sigma_{g,n}) \hookrightarrow \mathrm{Rep}_{\alpha}^{\mathrm{DT}}(\Sigma_{g,n}, \mathrm{PSL}_2 \mathbb{R})$

[M.]

↑ ergodic